Does an infinite Skolem sequence optimize anything?

Saad Mneimneh

June 2024

An infinite Skolem sequence $s_1, s_2, s_3, \dots$ is such that for every $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ there exist exactly two integers $a_n < b_n$ that satisfy $s_{a_n} = s_{b_n} = n$. Furthermore, $b_n - a_n = n$. For example, the first lexicographic Skolem sequence given by:

$$1, 1, 2, 3, 2, 4, 3, 5, 6, 4, 7, 8, 5, 9, 6, \dots$$

also satisfies $n < m \Leftrightarrow a_n < a_m$. Does an infinite Skolem sequence optimize anything?

For a given infinite sequence that contains every integer $n \in \mathbb{N}$ exactly twice, assume that the limits $\lim_{n\to\infty} a_n/n = \alpha$ and $\lim_{n\to\infty} b_n/n = \beta$ exist. It is not hard to see that $\alpha \geq 1$ and $\beta \geq 2$. First, pick an integer m. Let $n \leq m$ be such that all first occurrences of the integers $\{1, 2, \dots, m\}$ appear within the first a_n terms of the sequence. Obviously $a_n \geq m \geq n$. There are infinitely many such pairs (m, n) and, therefore, $\alpha \geq 1$. Similarly, let $n \leq m$ be such that all second occurrences of the integers $\{1, 2, \dots, m\}$ appear within the first b_n terms of the sequence. Then $b_n \geq 2m \geq 2n$. There are infinitely many such pairs (m, n) and, therefore, $\beta \geq 2$. In addition, $1/\alpha$ and $1/\beta$ can be thought of as the rate of adding new integers to the sequence, and the rate of adding second copies to the sequence, respectively. Therefore, we have $1/\alpha + 1/\beta = 1$.

Now, let's say that we want the first and second copies of every integer to appear as soon as possible. Obviously, the two extremes are $1, 2, 3, \dots$ with $\alpha = 1$ and β undefined, and $1, 1, 2, 2, 3, 3, \dots$ with $\alpha = \beta = 2$. This is shown in the figure below. We would like α to be close to 1 and β to be close to 2. Therefore, we can minimize $\max(\alpha - 1, \beta - 2)$. This can be done by making $\alpha - 1 = \beta - 2$ as shown. Combined with the constraint $1/\alpha + 1/\beta = 1$, this leads to $\alpha = \phi = (1 + \sqrt{5})/2$ (the golden ratio).

An infinite Skolem sequence satisfies $b_n = a_n + n$, so $\beta = \alpha + 1$ as desired. In fact, the first lexicographic infinite Skolem sequence has $a_n = \lfloor n\phi \rfloor$ and $b_n = a_n + n = \lfloor n\phi \rfloor + n = \lfloor n\phi + n \rfloor = \lfloor n(\phi + 1) \rfloor = \lfloor n\phi^2 \rfloor$.

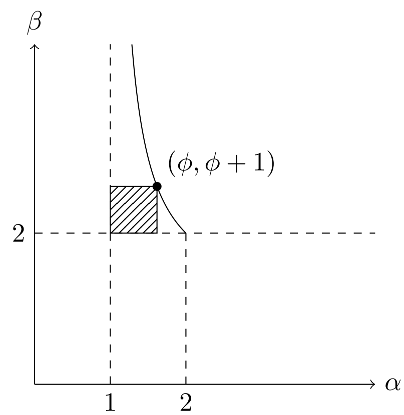

Figure 1: The constraint $1/\alpha + 1/\beta = 1$ with $\alpha \geq 1$ and $\beta \geq 2$. The point $(\phi, \phi + 1)$ minimizes $\max(\alpha - 1, \beta - 2)$ as shown by the square, making $\alpha - 1 = \beta - 2$.

Therefore, if we wish for both copies of every integer to appear as soon as possible, we can use the first lexicographic infinite Skolem sequence. Why is this useful? Well, I don't know. But imagine the following fictitious scenario. Every integer corresponds to a message, but for every integer n, there is a non-zero probability that message n is lost once, and a zero probability that message n is lost twice. One naive communication approach is to send an infinite stream where every message appears exactly twice. However, we want the recipient to receive messages $1, 2, \ldots, n$ as soon as possible for every n. It looks like the first lexicographic infinite Skolem sequence will achieve just that!

Note: The idea of seeing both copies of n as soon as possible is informal, because there are other formulations that could be interpreted as such. For instance, one might want to minimize $\max(1 - 1/\alpha, 1/2 - 1/\beta)$, or find a multiplier r such that $\alpha = 1/r$ and $r \cdot 1 + r \cdot 1/2 = 1$, which would result in different optimal values that an infinite Skolem sequence does not achieve. The goal was to show a non-trivial setting for which the infinite Skolem sequence is optimal.