Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

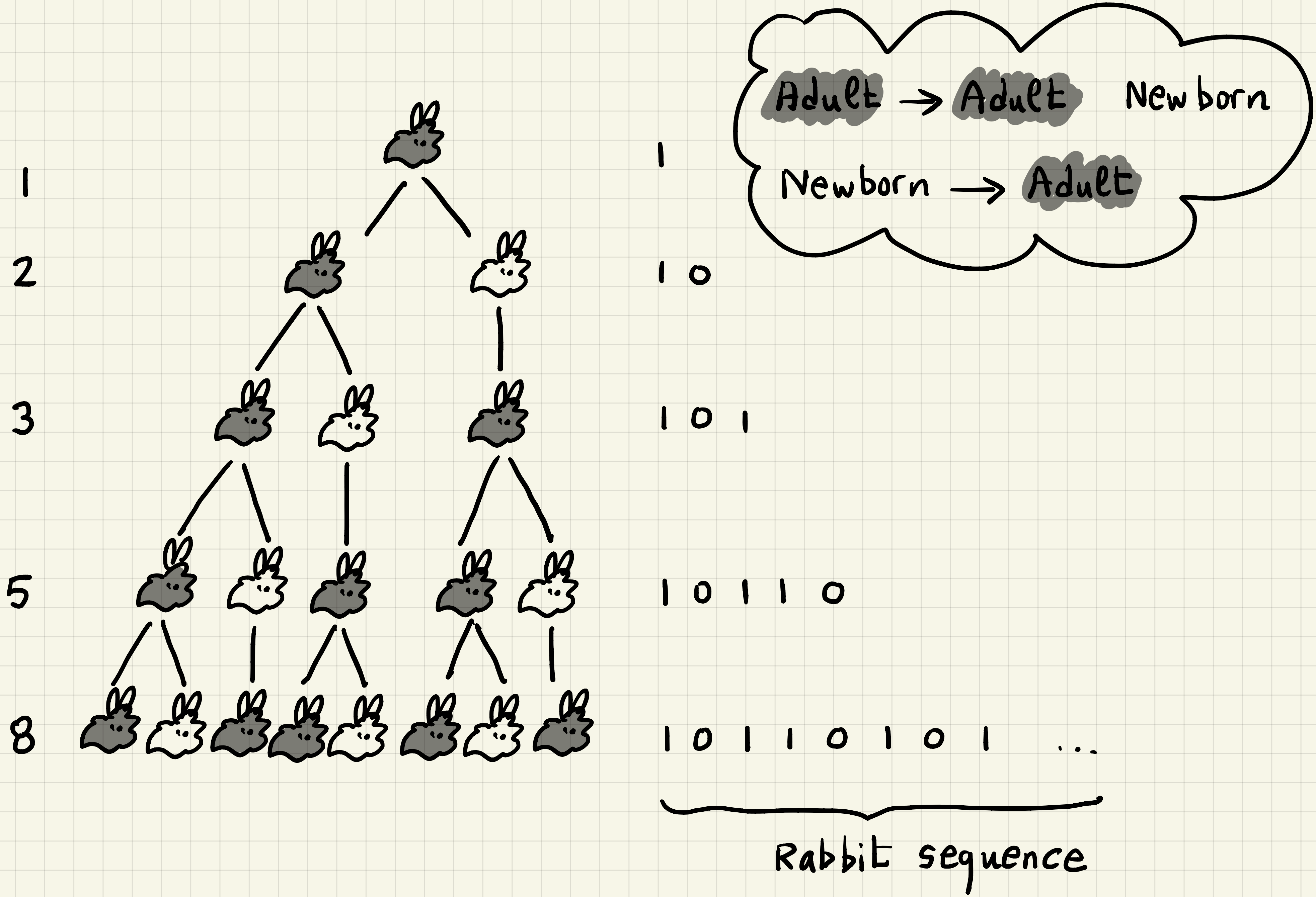
- $$F_n = \begin{cases} n & n \leq 1 \\ F_{n-1} + F_{n-2} & n > 1 \end{cases}$$

- $$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = 1.618\ldots \text{ (Golden ratio)}$$

∞, 1, 2, 1.5, 1.666, 1.6, 1.625, 1.615, 1.619, 1.617, ...

- $$x^2 - x - 1 = 0 \implies \begin{cases} x = \phi = \dfrac{1+\sqrt{5}}{2} \\ x = 1 - \phi \end{cases} \quad \left(\frac{1}{\phi} = \phi - 1 \right)$$

Don't forget about the rabbits



Problem 1: Skolem sequence

- A skolem sequence for integer n is a sequence of $2n$ integers $s_1, s_2, \dots, s_{2n}$ such that every k in $\{1, 2, \dots, n\}$ appears exactly twice at distance k.

- Example for $n = 4$: $\quad 1\ 1\ 3\ 4\ 2\ 3\ 2\ 4$

- Exists iff $n \equiv 0, 1 \pmod 4$

  (remainder in division by 4 is 0 or 1)

Make it easy $\Longrightarrow$ go to infinity!

- **Infinite skolem sequence** $s_1, s_2, s_3, \dots$ is such that <u>every</u> integer $n \geq 1$ appears exactly twice at distance n.

- Which infinite sequence? The **first** one:

$$n < m \iff n \text{ appears before } m$$

_	_	_	_	_	_	_	_	_	_	_	_	_	_	_
1	1	_	_	_	_	_	_	_	_	_	_	_	_	_
1	1	2	_	2	_	_	_	_	_	_	_	_	_	_
1	1	2	3	2	_	3	_	_	_	_	_	_	_	_
1	1	2	3	2	4	3	_	_	4	_	_	_	_	_
1	1	2	3	2	4	3	5	_	4	_	_	5	_	_
1	1	2	3	2	4	3	5	6	4	_	_	5	_	6
1	1	2	3	2	4	3	5	6	4	7	_	5	_	6
1	1	2	3	2	4	3	5	6	4	7	8	5	_	6
1	1	2	3	2	4	3	5	6	4	7	8	5	9	6

next number gets first available gap

The real challenge ...

- Let a_n be the position of the first n

$$1 \quad 1 \quad 2 \quad 3 \quad 2 \quad 4 \quad 3 \quad 5 \quad 6 \quad 4 \quad \ldots$$

$$a_1 = 1 \qquad a_2 = 3 \qquad a_3 = 4 \qquad a_4 = 6 \qquad a_5 = 8$$

- How do I find a_n ?
  - Let k be large enough (how large)
  - generate $s_1, s_2, \ldots, s_k$
  - See when n first shows up

_	_	_	_	_	_	_	_	_	_	_	_	_	_	_
1	1	_	_	_	_	_	_	_	_	_	_	_	_	_
1	1	2	_	2	_	_	_	_	_	_	_	_	_	_
1	1	2	3	2	_	3	_	_	_	_	_	_	_	_
1	1	2	3	2	4	3	_	_	4	_	_	_	_	_
1	1	2	3	2	4	3	5	_	4	_	_	5	_	_
1	1	2	3	2	4	3	5	6	4	_	_	5	_	6
1	1	2	3	2	4	3	5	6	4	7	_	5	_	6
1	1	2	3	2	4	3	5	6	4	7	8	5	_	6
1	1	2	3	2	4	3	5	6	4	7	8	5	9	6

2 1 2 2 1 2 1 2 …

$$a_2 - a_1 - 1 = 1$$
$$a_3 - a_2 - 1 = 0$$
$$a_4 - a_3 - 1 = 1$$
$$a_5 - a_4 - 1 = 1$$
$$a_6 - a_5 - 1 = 0$$
$$a_7 - a_6 - 1 = 1$$
$$a_8 - a_7 - 1 = 0$$
$$a_9 - a_8 - 1 = 1$$
$$\vdots \qquad \vdots$$
$$a_n - a_{n-1} - 1 = $$

rabbit sequence

$$a_n - a_1 - (n-1) = \sum_{i=1}^{n-1} b_i$$

$$a_n = n + \sum_{i=1}^{n-1} b_i$$

- Now, the sum $\sum_{i=1}^{n-1} b_i$ can be obtained by generating the rabbit sequence

Rabbit Rules

- Start with 10
- scan the bits

$1 \rightarrow 110$

$0 \rightarrow 10$

- Try : $10 \boxed{110\ 10\ 110 \ldots}$

- Better yet : $n + \sum_{i=1}^{n-1} b_i = \lfloor n\phi \rfloor$ (No need for array)

Example: $a_3 = \lfloor 3 \times 1.618 \rfloor = 4$

Generalizations

- What if 3 copies ?

  3 "blocked"

  $1\ 1\ 1\ 2\ \underline{3}\ 2\ \_\ 2$

  - $a_3 = 28$ $\quad$ a_n not monotonically increasing
  - Is a_n finite for every n ?

- Consider k copies but only last two must be at distance n.

  $k=3$: $1\,1\,1\,2\,2\,\underline{3}\,2\,3\,\underline{4}\,\underline{4}\,3\,\_\,\_\,4\ \ldots$

$$a_n = 1 + (k-1)(n-1) + \sum_{i=1}^{n-1} b_i$$

but

$$1 \to 1^k 0 = \overbrace{1\ldots1}^{k}0$$

$$0 \to 1^{k-1} 0 = \underbrace{1\ldots1}_{k-1}0$$