Partial order Relation

- Equivalence relation "groups" the elements
- Partial order relation "orders" the elements

Denote a partial order by $\prec$, so $a \prec b$ means $(a,b) \in R$

$\equiv$ to $=$ "is the same as" $\prec$ to $<$

1. Transitive. (as before)
2. Antisymmetric. $\forall a, b \in S,\ (a \prec b \wedge b \prec a) \Rightarrow a = b$
3. $\prec$ could be reflexive or not.

Example: $<$ on $\mathbb{R}$, $\leq$ on $\mathbb{R}$

(not reflexive) (reflexive)

If S is finite, then S admits a minimum

$$\exists e \in S, \forall x \in S, x \neq e \Rightarrow x \not< e$$

<u>proof:</u> Suppose e does not exist, I can find an infinite sequence

$$a_1 > a_2 > a_3 \cdots \qquad \text{where } a_i \neq a_{i+1}$$

Since S is finite, we must cycle

$$\cdots \overbrace{a_i > \cdots > a_j}^{\text{(transitivity)}} > \cdots > a_i \cdots$$

$$\left.\begin{array}{l} a_j < a_i \\ a_i < a_j \end{array}\right| \Rightarrow \text{Contradiction! (not antisymmetric)}$$

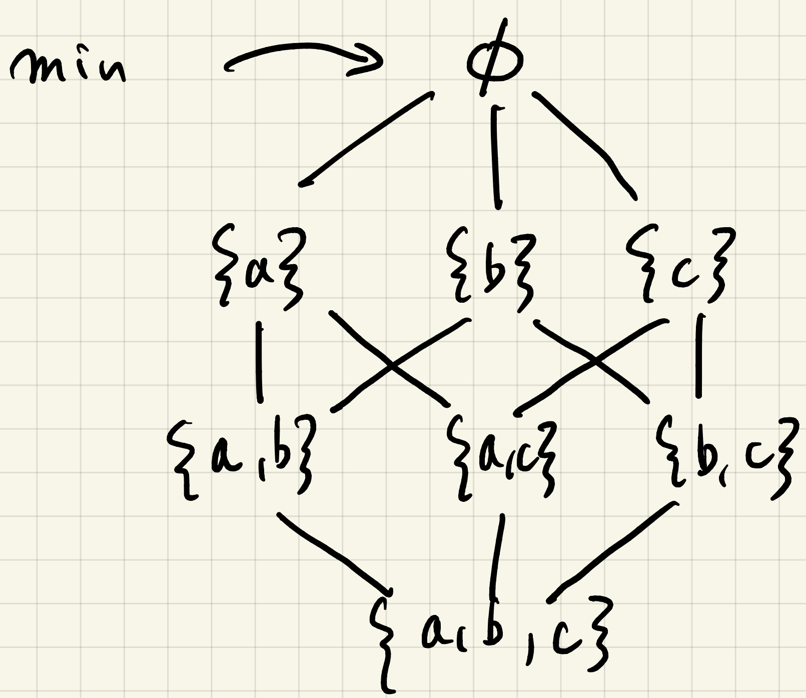
Example: $S = \{a, b, c\}$

$$\mathcal{P}(S) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$$

Relation: $X < Y \iff X$ is a proper subset of Y

Transitive. $X \subset Y \wedge Y \subset Z \Rightarrow X \subset Z$

Antisymmetry. $(X \subset Y \wedge Y \subset X) \Rightarrow X = Y$



"Hasse Diagram."

All edges that can be inferred by transitivity are omitted.

Example: $(a,b) \prec (c,d) \iff (a < c) \vee (a = c \wedge b < d)$

Exercice : Prove this is a partial order relation.

Transitive: $(a,b) \prec (c,d)$

$(c,d) \prec (e,f)$

1) $a < c \quad \wedge \quad c < e \implies a < e$

2) $a < c \quad \wedge \quad c = e \implies a < e$

3) $a = c \quad \wedge \quad c < e \implies a < e$

4) $(a = c \wedge b < d) \wedge (c = e \wedge d < f) \implies a = e \wedge b < f$

Therefore $(a,b) \prec (e,f)$

$$(a, b) \prec (c, d) \iff (a < c) \vee (a = c \wedge b < d)$$

<u>Antisymmetry</u>.

$$\left.\begin{array}{l}(a, b) \prec (c, d) \\ (c, d) \prec (a, b)\end{array}\right\} \text{ can't happen simultaneously}$$

1) $a < c \quad \wedge \quad c < a \qquad \times$
2) $a < c \quad \wedge \quad c = a \qquad \times$
3) $a = c \quad \wedge \quad c < a \qquad \times$
4) $b < d \quad \wedge \quad d < b \qquad \times$

Note: In general, to prove antisymmetry, prove

either: $x \prec y \ \wedge \ y \prec x \implies x = y$

or : $x \prec y \ \wedge \ y \prec x$ is false

Consider the following program in pseudocode where $x = \{...\}$ assigns x a value from the set, and $(x, y) = (..., ...)$ simultaneously assigns x and y their values:

```
(x,y,z)=({1,...,n},{1,...,n},{1,...,n})
while x>0 and y>0 and z>0
  control={1,2,3}
  if control==1 then
    (x,y,z)=(x+1,y-1,z-1)
  else
  if control==2 then
    (x,y,z)=(x-1,y+1,z-1)
  else
    (z,y,z)=(x-1,y-1,z+1)
```

$x+y+z$ decreases by 1 each iteration.

It is typical to prove that a program terminates by finding a quantity that is always decreasing. In the above program, obviously $x + y + z$ decreases by 1 after every iteration. Therefore, one of x, y, or z will eventually reach zero and the program will terminate. However, it is not always possible to find a decreasing quantity, like in the following program:

```
(x,y,z)=({1,...,n},{1,...,n},{1,...,n})
while x>0 and y>0 and z>0
  control={1,2}
  if control==1 then
    x={x,...,n}
    y={y,...,n}
    z=z-1
  else
    y={y,...,n}
    x=x-1
```

In each iteration
either z decreases,
or z remains the same
but x decreases.

Look at (z, x)

let z_i, x_i be values of z and x in iteration i

$$(z_i, x_i) \prec (z_j, x_j) \iff z_i < z_j \vee (z_i = z_j \wedge x_i < x_j)$$

Iteration i v.s Iteration $(i+1)$

$$(z_{i+1}, x_{i+1}) \prec (z_i, x_i)$$

because either $z_{i+1} < z_i$ or

$z_{i+1} = z_i \wedge x_{i+1} < x_i$

Finite set of possible tuples, every partial order relation on a finite set has a "minimum", we can't decrease (z, x) indefinitely. Program must stop.