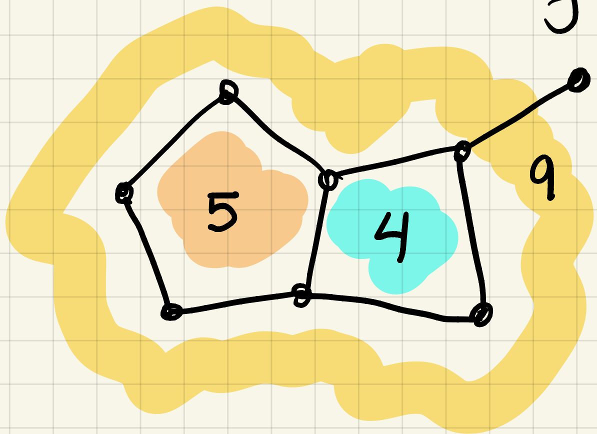
Which graphs are not planar ?

First, define degree of a face to be the number of edges on a closed walk of its boundary

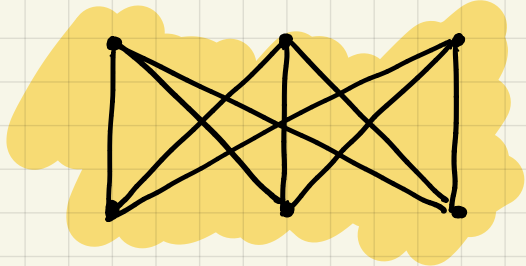


$5+4+9=18$

$e=9$

$$\sum_F d_F = 2e$$

$K_{3,3}$ is not planar

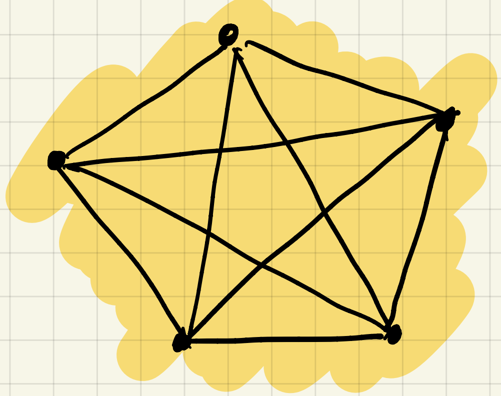


every face has at least 4 edges

$$4f \leq \sum_F d_F = 2e = 18$$

$v = 6,\ e = 9 \Rightarrow f = 5$, but $4 \times 5 > 18$

K_5 is not planar

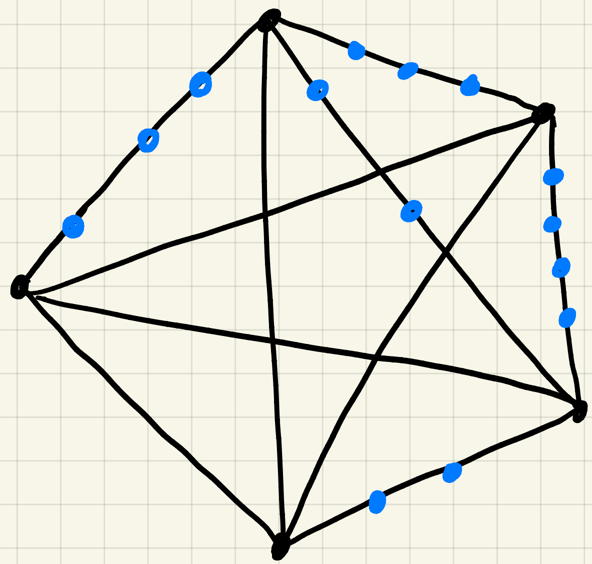


every face has at least 3 edges

$$3f \leq \sum_F d_F = 2e = 20$$

$v = 5,\ e = 10 \Rightarrow f = 7$, but $3 \times 7 > 20$

Every non-planar graph has $K_{3,3}$ or K_5 as a "basic shape" subdivision of a graph



planar $\Longleftrightarrow$ No subdivision of $K_{3,3}$ or K_5

Another interesting result:

Every planar graph with $v > 2$ satisfies

$$e \leqslant 3v - 6$$

proof: Since every face has degree at least 3 (because $v > 2$)

we have $3f \leqslant 2e \Rightarrow f \leqslant \frac{2e}{3}$

but $e = v + f - 2 \leqslant v + \frac{2e}{3} - 2 \Rightarrow e \leqslant 3v - 6$

Average degree

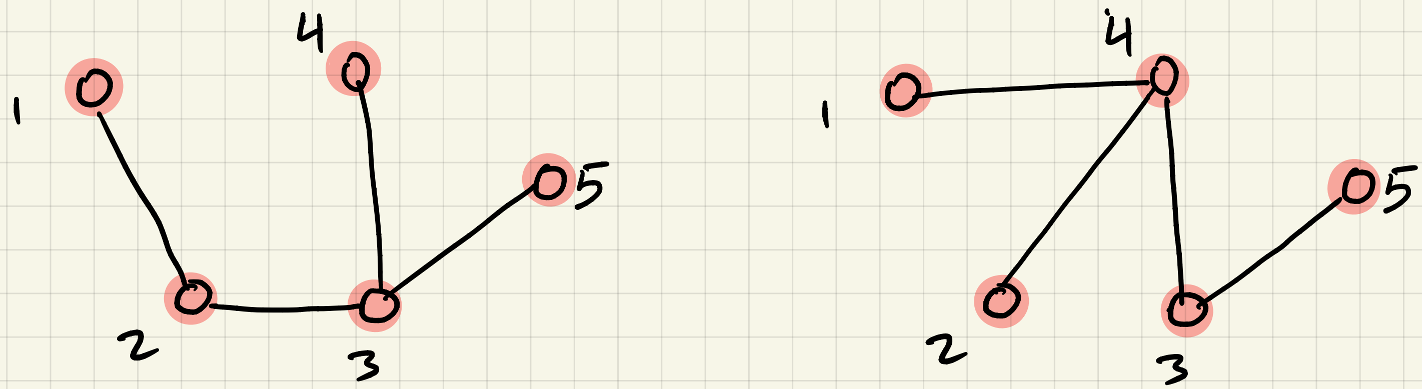
$$\sum_{v \in V} d_v = 2e \leqslant 6v - 12$$

$$\frac{\sum d_v}{v} \leqslant 6 - \frac{12}{v} < 6$$ (there must be a vertex v such that $d_v \leqslant 5$)

Number of Trees

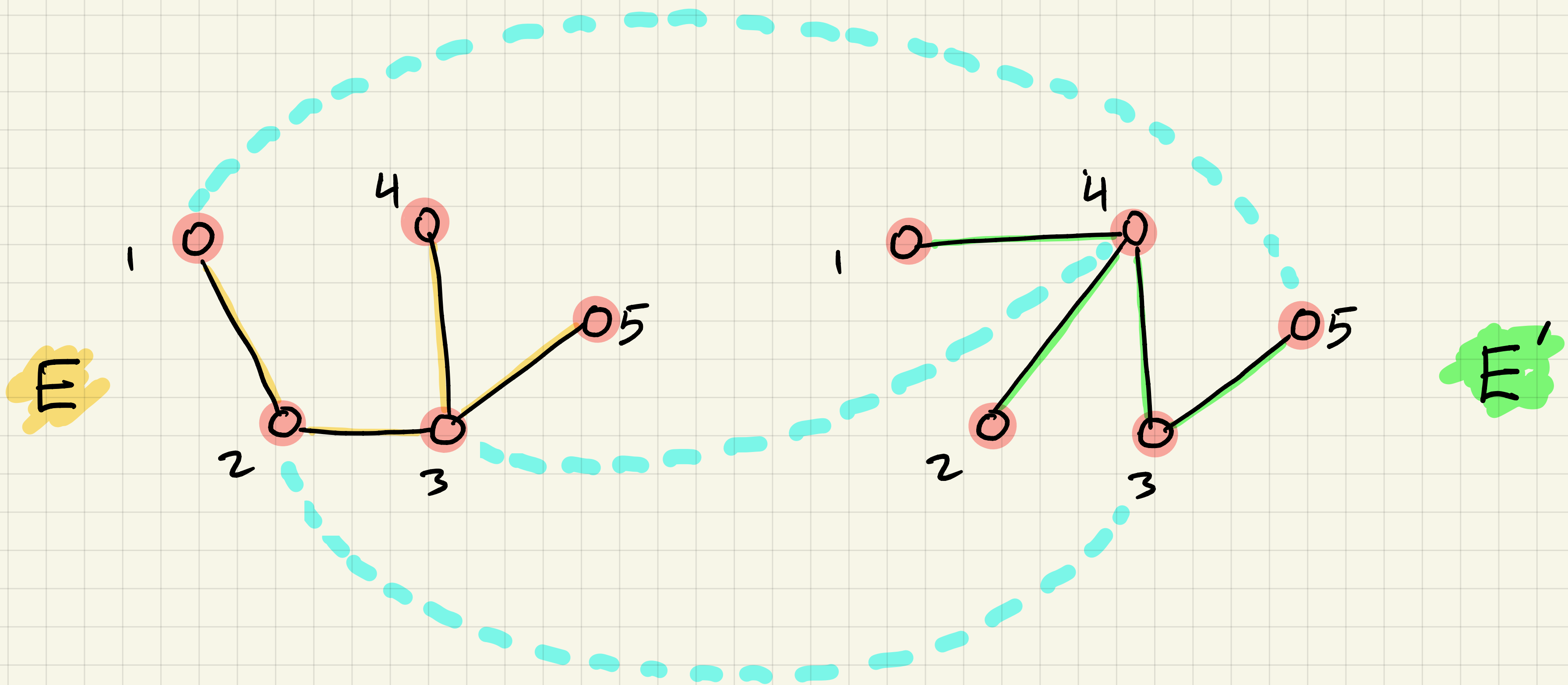
Given n vertices, how many trees can we make?

It depends! Are these the same?



Labeled trees: Two trees are the same if they have the same set of edges

Unlabeled trees: Two trees are the same if there exists a bijection between their vertices that preserves the edges.



$f(1) = 5 \quad f(2) = 3 \quad f(3) = 4 \quad f(4) = 1 \quad f(5) = 2$

$$(u, v) \in E \iff (f(u), f(v)) \in E'$$

Cayley's Formula:

Number of labeled trees on n vertices is n^{n-2}

Chapter 8 contains 3 proofs

1) Prufer code
2) Inclusion-Exclusion
3) A counting argument