CSCI 150 Fall 2024 TA's questions

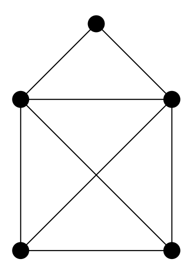
Tasmina, Vladislav, John, Randy,
Nicholas, Zach, Matthew, Zhen Tao
with help and edits by Saad
Computer Science, Hunter College

Week 1

Topics: The sum $T_n = 1 + 2 + \ldots + n = n(n+1)/2$ (triangular numbers), planar graphs, Euler's formula for planar graphs $v - e + f = 2$, T_{n-1} is the number of pairs on n objects, generalization of sum to $a + (a+s) + (a + 2s) + \ldots + b = \frac{a+b}{2}(\frac{b-a}{s} + 1)$, counting pairs, permutations and $n!$, sum and product notations $\sum$ and $\prod$, manipulation of sum and product notation, splitting sums and nested sums (also seen as nested loops), the addition rule.

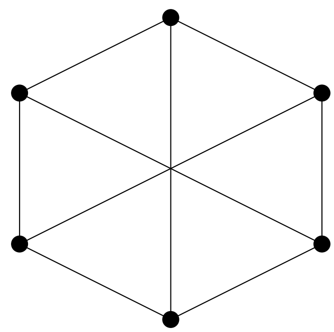
1. (Saad) Find a graph with 6 vertices and 9 edges that is not planar and a graph with 6 vertices and 9 edges that is planar. How many faces such a planar graph must have?

2. (Saad) Consider the following graph:

   Find what is wrong with this argument: If the graph is planar, then it should satisfy Euler's formula, which states that $v - e + f = 2$. Therefore, $5 - 8 + f = 2$ and $f = 5$. However, we see 6 faces (including the outer face). We conclude that this graph is not planar.

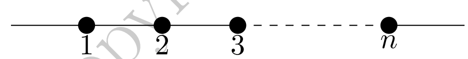
3. (Saad) Is the following graph planar?



4. (Saad) Assume that a and b are two integers such that $a \le b$. The interval $[a, b]$ contains the integers $a, a+1, \ldots, b$. How many integers are contained in $[a, b]$?

5. (Saad) Assume that we have a set containing all integers between 1 and n (inclusive) but it's missing one integer. What is a quick way to determine what the missing integer is?

6. (Saad) Consider n points on a line, as shown below:



How many segments of positive length can we make? How many segments of length at least 2 can we make? Can you generalize to a length of at least d?

7. (Saad) Is $T_n \le n!$?

8. (Saad) Is $n!$ always even?

9. (Saad) Complete this notation: $T_n = \binom{?}{?}$. Write $\binom{n}{2}$ in terms of factorials.

10. (Saad) Find a nice formula for the following sum:

$$1 + 2 + \ldots + (n-1) + \underbrace{n + n + \ldots + n}_{d} + (n-1) + \ldots + 2 + 1$$

11. (Saad) Express the above sum as (the expression inside the sum should be in terms of i and d)

$$\sum_{i=0}^{n-1}(\dots)$$

When $d = 1$, conclude something about the sum of the first n odd integers.

12. (Vlad) Prove the following equality: $T_n + T_{n-1} = n^2$. What does it mean geometrically?

13. (Matthew) A complete graph is an undirected graph with all nodes connected to every other node. Given a complete graph with n nodes, how many edges does it have? Can you think of an upper limit for n such that a complete graph will be planar?

14. (John) Consider $T_n = 1 + 3 + 5 + ... + (2n - 1)$. Using a geometric proof, find a formula that correctly represents this sequence. Afterward, use the generalized formula for counting sums to confirm whether your answer is correct. *Hint: How can you modify the geometric proof in class to find the answer?*

15. (Zhen Tao) A Pokemon can either be assigned a single type, or two unique types (Order of the type does not matter). Let's say the developers of Pokemon decided to introduce 5 new types on top of the existing 18, totaling 23 unique types. How many unique ways can you assign typing to a Pokemon?

16. (Nicholas Cheung) Can you find the first three triangular numbers that are perfect squares? *Hint: Using its explicit formula, when is T_n a perfect square?*

17. (Tasmina) You are walking in a forest, picking flowers, when you reach a fork that divides into left and right. In the left, the number of flowers increment by one every three feet. In the right, there is a dozen flowers at each increment of a foot. Assume n is the length of each fork (in feet) and that there are no flowers at the beginning of the fork. How many total flowers will you be able to pick on both paths? Simplify the answer using the principles discussed in class and identify which ones were used. *(Bonus: What would happen if there were no flowers at the end either?)*

18. (Saad) Compute the following expressions:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} 1 \qquad \sum_{i=1}^{n} \prod_{j=i+1}^{n} 1 \qquad \prod_{i=1}^{n} \sum_{j=i+1}^{n} 1 \qquad \prod_{i=1}^{n} \prod_{j=i+1}^{n} 1$$

19. (Saad) Which of the two is bigger (f is any function)?

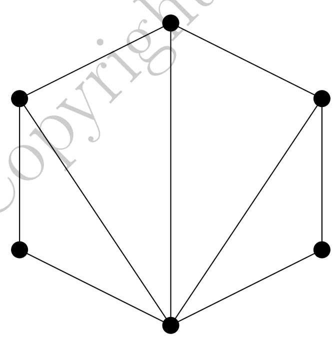
$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} f(i,j) \qquad \sum_{j=1}^{n} \sum_{i=1}^{j-1} f(i,j)$$

What about the following two?

$$\sum_{i=0}^{n} f(i) \qquad \sum_{i=0}^{n} f(n-i)$$

Solutions

1. Here's a planar graph with 6 vertices and 9 edges:



$K_{3,3}$ (houses and utilities) that we have seen in class has 6 vertices and 9 edges, and is not planar.

2. What's wrong with the argument is that the graph does not have 6 faces. Faces are only interpreted visually when the graph is drawn in a planar way. If the graph is planar, the number of faces must be $2 + e - v$, which is equal to 5 in this example. In fact, the graph is planar.

3. This graph is not planar. It is the same $K_{3,3}$ graph seen in class (houses and utilities) but drawn differently.

4. As explained in class, $b - a + 1$.

5. Add them up and subtract the result from $n(n+1)/2$.

6. There are $\binom{n}{2} = n(n-1)/2$ segments because each segment is defined by a pair of points. There are $(n-2)$ segments of length at least 2 with point 1 as the left point, $(1,3), \ldots, (1,n)$. Similarly, there are $(n-3)$ segments of length at least 2 with point 2 as the left point $(2,4), \ldots, (2,n)$. In total, and using the addition rule, there are $(n-2)+(n-3)+\ldots+1$ segments of length at least 2. This is $(n-2)(n-1)/2 = \binom{n-1}{2}$. In general, there are $(n-d)+(n-d-1)+\ldots+1$ segments of length at least d. This is $(n-d)(n-d+1)/2 = \binom{n-d+1}{2}$.

7. It is not always the case that $T_n \leq n!$; for instance, $T_2 = 3$ and $2! = 2$.

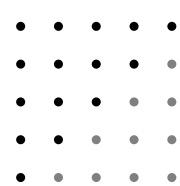
8. It is not always the case that $n!$ is even; for instance, $0! = 1! = 1$, which is odd.

9. $T_n = \binom{n+1}{2}$. $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n!}{2!(n-2)!}$.

10. We can rewrite the sum as $2[1+2+\ldots+(n-1)]+nd = 2n(n-1)/2+nd = n(n-1)+nd = n(n+d-1)$.

11. Based on the above, we have $2\sum_{i=1}^{n-1} i + \sum_{i=1}^{n} d = 2\sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} d = \sum_{i=0}^{n-1}(2i+d)$. Therefore, when $d = 1$, our sum is $\sum_{i=0}^{n-1}(2i+1) = 1+3+\ldots+(2n-1)$, which is the sum of the first n odd integers. By the formula found above, this is $n(n+1-1) = n^2$. So if we add up the first n odd integers, we get n^2.

12. Observe that $T_n + T_{n-1} = n(n+1)/2+(n-1)n/2 = (n^2+n+n^2-n)/2 = 2n^2/2 = n^2$. Here's one possible geometric interpretation (example for $n = 5$):

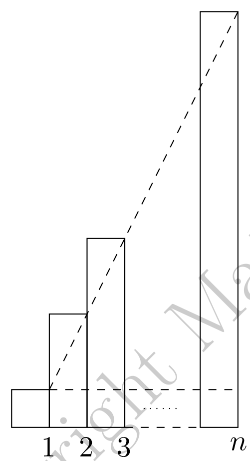


Black dots represent triangular number $T_5 = 1+2+3+4+5$ and gray dots represent triangular number $T_4 = 1+2+3+4$. We can see that $T_5 + T_4 = 5 \cdot 5 = 5^2$.

13. If you start with n nodes and no edges, you can pick a node and draw $n-1$ edges to connect it to another node. You can then choose a different

node and draw all edges to connect it with another node, and you will notice that you could only draw $n-2$ edges. Repeating this process, your result is $(n-1)+(n-2)+\ldots+1$, which is exactly $\frac{n(n-1)}{2}$. The upper limit for a complete graph to be planar is 4, because the complete graph on 5 vertices, called K_5, is not planar (and therefore any larger complete graph is not, since it contains K_5).

14. Consider the following illustration:



The total area of the bars is $1+3+\ldots+(2n-1)$ and consists of the lower rectangle of length n and height 1, the big triangle of base $n-1$ and height $2n-2$, and $n-1$ small triangles of base 1 and height 2. Therefore, the same area is equal to $n+(n-1)(2n-2)/2+(n-1)2/2 = n+(n-1)(n-1)+n-1 = 2n-1+(n-1)^2 = 2n-1+n^2-2n+1 = n^2$. Now, using the general formula for sums, we have $a=1$, $b=2n-1$, and $s=2$. We get

$$\frac{a+b}{2}\Big(\frac{b-a}{s}+1\Big)=\frac{2n}{2}\Big(\frac{2n-2}{2}+1\Big)=n(n-1+1)=n^2$$

15. There are 23 single types + $\binom{23}{2}$ dual types, so 276 ways (addition rule since the categories are disjoint).

16. Let $T_n = k^2$. This means $n(n+1)/2 = k^2$. This gives a quadratic equation: $n^2+n-2k=0$. The positive solution is given by:

$$n=\frac{-1+\sqrt{1+8k^2}}{2}$$

All we have to do now is find values of k that will make $1+8k^2$ a perfect square. This works for $k = 0$ and $k = 1$, to obtain $n = 0$ and $n = 1$, respectively. The next k that works is $k = 6$, which gives $n = 8$. Therefore, we have $T_0 = \sum_{i=1}^{0} i = 0 = 0^2$, $T_1 = \sum_{i=1}^{1} i = 1 = 1^2$, and $T_8 = \sum_{i=1}^{8} i = 36 = 6^2$.

17. In the left, the number of flowers increment by one at every 3 feet, which means that there are $n/3$ increments in total. Thus, you can write this as:

$$0 + 1 + 2 + 3 + \ldots + \frac{n}{3}$$

Notice that this looks similar to the formula $1 + 2 + 3 + \ldots + n$, which is equal to $n(n+1)/2$. Substituting n in this formula for $n/3$ yields:

$$\frac{\frac{n}{3}(\frac{n}{3}+1)}{2},$$

which represents the total number of flowers in the left path. In the right, there are a dozen flowers at each increment of 1 foot. Therefore, we simply have $12n$ flowers in total on the right path. Since the two paths are disjoint, the addition rule can be applied to combine them, yielding the total number of flowers to be picked as

$$\frac{\frac{n}{3}(\frac{n}{3}+1)}{2} + 12n$$

Bonus: If there were no flowers at the end, then the left path would have $1 + 2 + 3 + \ldots + (n/3 - 1)$ flowers and the right would have $12(n-1)$ flowers, which would add up to

$$\frac{(\frac{n}{3}-1)\frac{n}{3}}{2} + 12(n-1)$$

flowers.

18. $\binom{n}{2}, n, 0, 1.$

19. Neither. They are equal to each other. One sum goes through all i and all $j > i$, and one sum goes through all j and all $i < j$. The answer is the same of the second part. Both summations compute $f(0) + \ldots + f(n)$ but in different orders.

Week 2

Topics: Addition rule, multiplication rule, 4 kinds of selection: unordered without repetition, ordered without repetition, unordered with repetition, ordered with repetition, k-permutations, k-combinations, problems with binary strings, problems with words, handshake lemma, counting in general...

1. (Matthew) How many 7 digit integers are there such that all the digits are unique? How many of these have their digits in decreasing order?

2. (Randy) The degree of a vertex is the number of edges connected to it. A graph is connected if there is a path of edges between every pair of vertices. Construct a connected graph with all vertices having degree 3. What else can you say about your graph? Is there a special property that it must satisfy?

3. (Zach) You booked a vacation to Hawaii with a group of friends from high school. You have a lot of friends, and you all decided to book a flight on the same plane, but unfortunately because there are 15 of you, not all of you will fit on the plane (there are only 7 seats left). Everyone agrees to hold a lottery to see who gets to be on the flight, so you write down on separate pieces of scrap paper all the possible combinations of people who can go from your friend group (the maximum number allowed per combination). You will randomly pick one of the scraps of paper and the names that are written on the paper is who will go. How many pieces of scrap paper will be in the lottery?

4. (Tasmina) There are 5 food carts near Hunter. You want to go to a different one every day. How many different ways can you do so?

5. (Tasmina) How many ways can you rearrange the letters in my first name? Construct a procedure by which you can apply the product rule and identify where (and how much) the overcount is.

6. (Saad) This exercise is designed to highlight a potential confusion about overcounting when using the product rule. Consider the following two questions:

   (a) How many anagrams of "saad" are there?

   (b) How many words of length 2 can we make using the alphabet $\{a, b, c\}$ (letters can repeat)?

7. (Saad) In how many ways can we place one snake and one ladder on a chessboard if the head and tail of the snake must be on the same color. The snake head must still be higher than its tail, and the snake's head, tail, the ladder's bottom, and top must all be on different squares. As before, assume that the board size (number of squares) n is even.

8. (Zach) How many anagrams of my name, "ZachAry", are there if (the following conditions are independent, each is treated as a separate question):

   (a) The upper/lower case as shown for each letter is important?

   (b) 'Z' must be exactly in the middle, and upper/lower case is ignored?

   (c) Except for the second occurrence of 'a', all letters must appear in alphabetical order? Consider two scenarios: upper/lower case is ignored, and upper/lower case is important.

9. (Nicholas) Suppose that Saad walks into his office one day with "MATHMATH" written on his wall. In how many ways can Saad erase four of the letters so that the remaining letters spell out "MATH" in that order? *Hint*: Consider at what point in time you jump into the second set of letters. Can you do the same for "TEETHTEETH"? *Hint*: Consider which two 'E's to use, and in each case, list all possibilities, then use the addition rule.

10. (Vlad) A tetris board is 20 blocks high and 10 blocks wide. How many ways can you place the "L" piece anywhere on the board, assuming that you're allowed to translate the piece on the block grid or rotate it 90 degrees? *Hint*: Divide the board into groups of squares such that squares in one group allow the same amount of placements of the piece. Then use the addition rule.

11. (Zhen Tao Pan) A piano has 88 keys, with 52 white keys and the rest black keys.

    (a) Saad want to pick 10 unique keys for a music piece. How many ways can he choose?

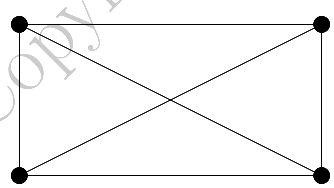
    (b) If he has to use equal amount of black and white keys for the 10 keys, how many ways can he choose?

    (c) For the 10 keys that he chose, if he could only play each key once, how many ways can he play those keys?

Solutions

1. This problem relates to permutations. There are 10 digits in total, and you want to form a 7 digit integer. $\frac{10!}{(10-7)!}$ gives you the total amount of 7 digit sequences with unique integers, but this does not account for 0 being the start (so that sequence is an invalid 7 digit integer). However, we can calculate the amount of 7 digit sequences that begin with 0 and subtract that value from $\frac{10!}{(10-7)!}$. The number of 7 digit sequences that begin with 0 would be $\frac{9!}{(9-6)!}$. Since we have 0 in the start, there are only 9 digits to choose from and 6 more digits to choose. Our result turns out to be $\frac{10!}{(10-7)!} - \frac{9!}{(9-6)!}$, which is 544320. If we begin to think about having the digits in decreasing order, this is no longer a permutation problem, but rather a combination problem. No matter what 7 digits we choose, we can always sort it into decreasing order. For example, if we have 9876543 and 5798364, the second integer is exactly the first integer if we were to rearrange the digits in decreasing order. Thus, this question is essentially asking how many combinations of 7 digits are there, which is $\binom{10}{7}$.

2. One possibility is the complete graph on 4 vertices (that's the smallest such graph).



   But this is not the only solution, there are many graphs that are 3-regular and connected; for instance, $K_{3,3}$. In general, you can make a cycle of $2n$ vertices, and then add n "chords" (each connecting a distinct pair of vertices) to obtain a 3-regular connected graph. This construction works for all $n > 1$. The graph must have an even number of vertices. In addition, the number of edges must be a multiple of 3. By the handshake lemma: $3v = 2e$, where $v = 2n$ for some $n > 1$. So, these are the possibilities for (v, e): $(4, 6), (6, 9), (8, 12), (10, 15), \ldots$.

3. The number of pieces of scrap paper in the lottery is $\binom{15}{7}$ because you're writing down all the combinations of 7 of the 15 names, each combination on a separate piece of scrap paper. This is 6,435.

4. First, you need to identify what type of selection this is. We know there is no repetition because the question explicitly states that you want to go to a different cart each day. However, there is no explicit statement on order. To determine this, you need to think about what it would look like if there was order versus no order. With order, a selection of ABCDE would be different from BCDEA, while without order, they would be identical. Because a cart is paired with a day (i.e. cart A is visited on Monday), order matters because you would visit different carts per day in the aforementioned examples. Thus, the number of ways you can visit each cart is
$$\frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!$$
Notice that this is equivalent to 5!, which you could have derived without figuring out order/repetition by understanding that this problem is one of permutations.

5. This is just anagrams. I have 7 letters, 2 of which repeat, so I can write this as $\frac{7!}{2!} = 2520$. The idea is the following:

   1. choose a position for the first letter ... 7 ways
   2. choose a position for the second letter ... 6 ways
   3. choose a position for the third letter ... 5 ways
   4. choose a position for the fourth letter ... 4 ways
   5. choose a position for the fifth letter ... 3 ways
   6. choose a position for the sixth letter ... 2 ways
   7. choose a position for the seventh letter ... 1 way

   By the product rule, this is 7!. However, since the second and seventh letters are the same, switching the choices in the corresponding phases (phase 2 and phase 7) results in the same outcome. This represents an overcount by 2.

6. To answer the first question, consider the following procedure consisting of 4 phases:

   1. Pick a letter from "saad" to be the first letter ... 4 ways
   2. Pick another letter from "saad" to be the second letter ... 3 ways
   3. Pick another letter from "saad" to be the third letter ... 2 ways
   4. Pick another (the last remaining) letter from "saad" to be the fourth letter ... 1 way

   By the product rule, we have $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ anagrams. However, there is overcounting here. If the <u>second letter</u> of "saad" is picked in phase i, and the <u>third letter</u> of "saad" is picked in phase j, then switching those <u>different</u> choices would still result in the same anagram. So we are overcounting by $2! = 1$. The correct answer is therefore $4!/2 = 12$.

   To answer the second question, consider the following procedure consisting of 2 phases:

   1. Pick any letter from the given alphabet $\{a, b, c\}$ to be the first letter of the word... 3 ways
   2. Pick any letter from the given alphabet $\{a, b, c\}$ to be the second letter of the word ... 3 ways

   By the product rule, we have $3 \cdot 3 = 9$ possible words. In fact, here they are $\{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$. There is no overcounting. But what about the following argument: If the letter 'a' is picked in the first phase, and the letter 'a' is picked in the second phase, then switching those choices would result in the same outcome (which is the word aa). So there must be overcounting. This argument is not correct, because in this case, the letter 'a' is the same physical object we are selecting from $\{a, b, c\}$ in both phases (because we allow repetition). Therefore, "switching" here does not correspond to making <u>different</u> choices. Overcounting occurs when making different choices results in the same interpretation of the outcome.

7. Consider the following procedure:

   1. Pick a square for the snake ... n ways
   2. Pick another square for the snake of the same color as above ... $n/2-1$ ways
   3. Pick another square for the ladder ... $n-2$ ways
   4. Pick another square for the ladder ... $n-3$ ways

   By the product rule, we have $n(n/2-1)(n-2)(n-3)$. Observe that the outcomes $(a,b,c,d),(b,a,c,d),(a,b,d,c),(b,a,d,c)$ are all equivalent (they defined exactly the same snake and ladder). Therefore, there is an overcounting by 4. So the correct answer is $n(n/2-1)(n-2)(n-3)/4$.

   What would go wrong if we consider the following procedure instead?

   1. Pick a square for the ladder ...
   2. Pick another square of the ladder ...
   3. Pick another square for the snake ...
   4. Pick another square for the snake of the same color as above ...

8. (a) In this case, each letter is distinct, so the answer is the number of ways we can permute the letters in my name: $7! = 5040$ anagrams.

   (b) In this case, we consider anagrams of "achary", and then simply place 'z' in the middle. As seen in previous questions, there are $6!/2!$ anagrams of "achary" because 'a' appears twice. The answer is therefore 360 anagrams.

   (c) First, assume the upper/lower case is ignored. Except for the second 'a', the anagram will be "achryz". There are 7 ways of placing the second 'a', two of which result in the same word. So we have 6 anagrams. If the upper/lower case is important, then each of the two 'a's ('a' and 'A') can appear first, giving 12 anagrams.

   Here's an extra idea: What if both 'a's are unconstrained in terms of the alphabetical order? Then we have to choose 2 positions among 7 to place the 'a's. If upper/lower case is ignored, the two positions are unordered. If, on the other hand, the upper/lower case is important, the two positions are ordered. The answer is then $\binom{7}{2}$ and $2\binom{7}{2}$, respectively.

9. The number of ways we can end up with “MATH” is given by when we jump to the second set of letters. We can jump immediately, after the first ’M’, after the first ’A’, after the first ’T’, or after the first ’H’. So, there are 5 ways:

$$----MATH$$

$$M----ATH$$

$$MA----TH$$

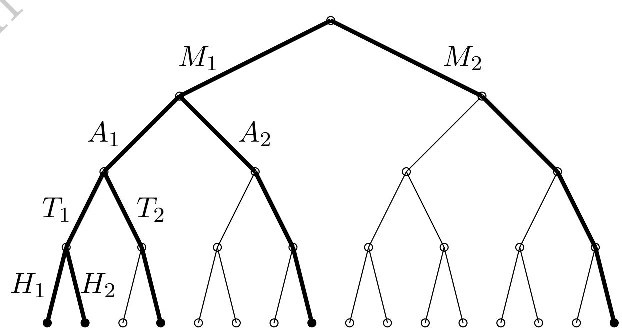
$$MAT----H$$

$$MATH----$$

Observe that the product rule based on the following procedure does not work:

1. choose one of the two Ms ... 2 ways
2. choose one of the two As ... 2 ways
3. choose one of the two Ts ... 2 ways
4. choose one of the two Hs ... 2 ways

This would result in $2^4 = 16$, which is wrong. The problem here is that, depending on what choices have been made in phase i, phase $i+1$ cannot be done necessarily in 2 ways. Here’s a decision tree showing the five outcomes and illustrating the dependence among phases. For instance, if the second M was chosen (M_2 as shown below) in the first phase, then the second phase (as well as all subsequent phases) can only be done in one way.



To find the number of ways we can end up with “TEETH”, let’s consider which 2 ’E’s are used. There are $\binom{4}{2} = 6$ possibilities for the ’E’s. In each, we can figure out how many ways we can make "TEETH", then add them up by the addition rule, since each category will have a different pair of 'E's (they are disjoint).

$$T\mathbf{EE} - - - - - - - - \ (3)$$

$$T\mathbf{E} - - - - \mathbf{E} - TH \ (1)$$

$$T\mathbf{E} - - - - - - \mathbf{E}TH \ (1)$$

$$T - \mathbf{E} - - - \mathbf{E} - TH \ (1)$$

$$T - \mathbf{E} - - - - \mathbf{E}TH \ (1)$$

$$- - - - - - - \mathbf{EE}TH \ (2)$$

We have $3 + 1 + 1 + 1 + 1 + 2 = 9$ ways.

10. Let the width be the X axis and the height be the Y axis of the board, with the bottom left corner of the board having coordinates (1,1). Let the coordinate of the L piece be the x and y position of the corner square of the piece marked by * like so:

   @
   @
   *@

   The piece has 4 orientations, one for each 90-degree rotation. Let's consider the orientation pictured above first:
   The X coordinate of the piece can have 9 values: 1 to 9.(it can't be at x=10 because then the block to the right of the corner would be out of bounds)
   Similarly, the Y coordinate can have 18 values: 1 to 18.
   9*18 = 162 ways to place L piece in the first orientation.

   Rotate the piece 90 degrees clockwise.
   Now the X coordinate can have 8 values: 1 to 8.
   The Y coordinate can have 19 values: 2 to 20.
   8*19 = 152 ways to place the L piece in the second orientation.
   Similarly, we can find the number of ways to place the piece for the other 2 orientations for a total of:
   162 + 152 + 162 + 152 = 628 ways

11. $\binom{88}{10}$, $\binom{52}{5}\binom{88-52}{5}$, 10!.

Week 3

Topics: Four ways of selection, tuples, sets, multisets, power set, functions, onto, one-to-one, bijection. Note: we will use the notation $\left(\!\binom{n}{k}\!\right)$ to represent the number of unordered selections with repetition (counting multisets).

1. (Vlad) Let S be the set of all satellites in the solar system and P be the set of all planets in the solar system. Define $f : S \to P$ such that $f(s) = p$ means "satellite s is orbiting planet p". Is this function one to one, onto, both or neither? If you could change the solar system, what would you have to change (if anything) to make f a bijection?

2. (Saad) Let S and T be two finite sets. Write down a formula relating the following 4 quantities:

$$|\mathcal{P}(S)| \quad |\mathcal{P}(T)| \quad |\mathcal{P}(S \cap T)| \quad |\mathcal{P}(S \cup T)|$$

3. (Saad) Consider the problem of distributing 3 gifts among 5 kids $\{A, B, C, D, E\}$. In how many ways can we do this given the following 4 scenarios:

   - all gifts are different, and each kid can receive at most 1 gift
   - all gifts are different, and kids can receive multiple gifts
   - all gifts are identical, and each kid can receive at most 1 gift
   - all gifts are identical, and kids can receive multiple gifts

   For each scenario, figure out whether the outcome can be encoded with tuples, sets, or multisets, and based on that compute the number of ways this can be done.

4. (Saad) This is a similar problem to the above: Find the number of ways of placing 4 marbles in 10 distinguishable boxes if:

   - The marbles are distinguishable, and no box can hold more than one marble.
   - The marbles are indistinguishable, and no box can hold more than one marble.
   - The marbles are distinguishable, and each box can hold any number of them.
   - The marbles are indistinguishable, and each box can hold any number of them.

5. (John) An accomplished mage was invited to a prestigious library where he was presented with 7 magic grimoires to keep. However, 3 of these grimoires contain forbidden spells, and so he is only allowed to keep at most one of them. What is the total number of combinations of grimoires the mage can keep?

6. (Matthew) A small neighborhood with 7 large homes can hold 47 people. Let h_i denote home i. Given that h_1 must hold at least 4 people, h_2 must hold at least 5 people, and h_3 must hold at least 2 people, how many ways are there to house all 47 people?

7. (Randy) Suppose that X and Y are finite sets. How many functions $f : X \to Y$ exist? What if they have to be one-to-one (injections) or bijections?

8. (Nicholas) This is a continuation of Matthew's problem in Week 2. How many 7 digit integers are there given that digits can be repeated and must be written in non-increasing order?

9. (Zach) The New York Yankees are about to start the postseason, but their manager forgot the positions for each player! Uh oh! There are 9 players and 9 positions. Can we construct a function from this information to help remind him? If so, does this function have any special properties? If so, what are they and how come? If such a function exists, what are the domain and co-domain of said function?

10. (Zhen) In statistics, an event is a subset of a set of outcomes. Consider rolling a 6-sided die labeled 1-6. The set of outcomes is $\{1, 2, 3, 4, 5, 6\}$.

    (a) How many events are there?

    (b) Consider rolling the die n times. The set of outcomes is now S^n. How many outcomes correspond to all rolls are odd? What about even? What about alternating (even, odd, even, ...) or (odd, even, odd, ...)?

11. (Tasmina) You are at an auction where 30 art pieces are being sold. You have your heart set on getting these 2 specific items, but are amenable to taking more pieces. How many different ways can you acquire art pieces? (Hint: Think of sets, rather than ways of selection.)

Solutions

1. The function is neither.
   It is not one to one because some planets(For example Jupiter) have multiple satellites, so $f(\text{Io}) = f(\text{Europa}) = \text{Jupiter}$, but $\text{Io} \neq \text{Europa}$.
   It is not onto because some planets in the Solar system have no satellites. For example, there is no satellite s such that $f(s) = \text{Mercury}$.
   To make f a bijection we would have to make sure that each planet has exactly one satellite orbiting it.

2. We know that $|S \cup T| = |S| + |T| - |S \cap T|$. Therefore,

   $$2^{|S\cup T|} = 2^{|S|+|T|-|S\cap T|} = \frac{2^{|S|}2^{|T|}}{2^{|S\cap T|}}$$

   So

   $$\mathcal{P}(S)\mathcal{P}(T) = \mathcal{P}(S \cap T)\mathcal{P}(S \cup T)$$

3. In the first scenario, we can encode each outcome as a tuple, with no repetition; for instance, (A, D, E) means that A receives gift 1, D receives gift 2, and E receives gift 3. There are $5!/(5-3)!$ such tuples. This is a 3-permutation of 5 people. This is equivalent to selecting 3 out of 5 with order and no repetition.

   In the second scenario, it's also a tuple, but with repetition. For instance, (A, B, A) means that A receives gift 1, B receives gift 2, and A receives gift 3. There are 5^3 such tuples. This is equivalent to selecting 3 out of 5 with order and repetition.

   In the third scenario, it's a set. For instance, $\{A, B, C\}$ means that A, B, and C each receive a gift. There are $\binom{5}{3}$ such sets. This is equivalent to selecting 3 out of 5 (no order and no repetition), a 3-combination of 5 people.

   Finally, in the last scenario, it's a multiset. For instance, $\{A, A, C\}$ means that A receives two gifts. and C receives one gift. There are $\left(\!\binom{5}{3}\!\right)$ such multisets. This is equivalent to selecting 3 out of 5 with repetition but no order.

4. $10!/(10-4)!$, $\binom{10}{4}$, 10^4, $\left(\!\binom{10}{4}\!\right)$ (we will see what this equals to next time).

5. Let us first calculate the number of combinations without the forbidden book. This equals $|\mathcal{P}(S)| = 2^4$ where S is the set of non-forbidden

grimoires. To calculate the number of combinations involving the forbidden book, we first calculate the number of ways to involve said book, which is $\binom{3}{1}$. Using the product rule, we compute $\binom{3}{1} * 2^4 = 48$. Finally, sum the two disjoint sets (number of combinations with and without the forbidden grimoire) to get $16 + 48 = 64$ ways.

6. This scenario can be modeled by the equation

$$(x_1 + 4) + (x_2 + 5) + (x_3 + 2) + x_4 + x_5 + x_6 + x_7 = 47$$

where x_i represents the number of people in the ith house. In order to find out the total amount of solutions for this equation, we have to first move to constants to the other side. We will then get

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 36$$

Now, we can apply stars and bars. For this equation, we have 36 stars and 6 bars. We can then do $\binom{36+6}{6}$, so there are $\binom{42}{6}$ ways to house the 47 people. Alternatively, you can think of this kind of selection as unordered with repetition, so you apply the formula $\binom{n+k-1}{k-1}$. Plugging in 36 for n and 7 for k, we also get $\binom{42}{6}$.

7. Let $n = |X|, m = |Y|$. There are m^n functions because for each $x \in X$ there are m choices of function values.
If $n > m$ then there are no injections (*why?*). Otherwise there are $\binom{m}{n}n!$ injections because we select n distinct values from Y and for each of those sets of values we have $n!$ ways to assign them to members of X.
If $m \neq n$ there are no bijections (*why?*). Otherwise there are $n! = m!$ bijections because a bijection can be represented as a n length sequence of values from Y with the ith element of X mapping to that value in Y. In both cases, we have a n-permutation of m things, i.e. this is $P(m, n)$, where $m \geq n$. It can be seen by the following procedure (illustrating the product rule).

1. Pick an element $y \in Y$ for $x_1 \in X$ (so that $f(x_1) = y$) ... m ways
2. Pick another element from Y for $x_2 \in X$ ... $m - 1$ ways
...
n. Pick another element from Y for $x_n \in X$ ... $m - n + 1$ ways

By the product rule, we have $m \times (m-1) \times \ldots \times (m - n + 1) = m!/(m - n!) = \binom{m}{n}n!$. When $m = n$, this is $m! = n!$.

8. Given that we're dealing with repeated digits and that the digits need to be ordered in a certain way, this problem relates to multisets. We have 7 total digits and 10 numbers (0-9) to choose from for those digits. The solution to this problem is thus equivalent to the number of integer solutions to:

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 7$$

   where x_i represents the number of times i appears. There is one edge case, however, that we have to worry about, and that is when we pick 7 zeros. Thus, our actual solution is $\left(\!\binom{10}{7}\!\right) - 1 = \binom{16}{9} - 1 = 11439$.

   The subtract 1 in this question comes from the fact that we can have at most 6 zeros to form a valid 7 digit integer, and there is one choice of digits that invalidates this restriction. While it is trivial for this question to find the number of invalid choices, as further exploration, can you identify a more general strategy to count the number of multisets with the restriction being an upper limit for some number of variables?

9. We can construct a function where Players is the domain and Positions is the co-domain. This function has the properties that it is one-to-one and onto (and thus a bijection). It is one-to-one because each position can be occupied by at most one player, and it is onto because each position is guaranteed to be assigned to a player. So such function is also necessarily a bijection.

10. 2^6, 3^n, 3^n, 3^n (for each roll in the outcome, there are three choices).

11. This should remind you of problem 6 from HW 2. You are given a set of art pieces, where the set is of size 30. You must find the number of possible subsets (otherwise known as the size of the power set) with the added conditions that 2 art pieces must be included. The identities of these art pieces don't matter. All that matters is that you know they must be included, so the possible art pieces you can choose are from the remaining 28 in the set. Using the power set, this means that the answer is 2^{28}.

Week 4

Four ways of selection, functions with properties one-to-one, onto, bijection, counting by establishing bijections, binomial coefficient properties, binomial theorem (and multinomial theorem), anagrams.

1. (Saad) Consider the following two sets, where n is a given positive integer:

$$S = \{(i,j)|i,j \in \mathbb{N} \text{ and } i < j \leq n\}$$

$$T = \{(x,y)|x,y \in \mathbb{N} \text{ and } x + y \leq n\}$$

   - Show that $|S| = |T|$ by establishing a bijection. (You will need to come up with the function $f : S \to T$, and show that f is both one-to-one and onto.)
   - Find $|T|$ by finding $|S|$ first, which can be done by identifying what S really represents.
   - Another way to find $|T|$ is by imagining a third variable $z \geq 0$, such that $x + y + z = n$, and counting solutions to the equation (what are the proper constraints?).
   - Yet another way to find $|T|$ is by considering all (x,y) such that $x + y = n$, and all (x,y) such that $x + y = n - 1$, ..., and all (x,y) such that $x + y = 2$. Finally, we use the addition rule.

   Note: This would have been an excellent test question, but it's too late now...

2. (Vlad) Let L be the set of all english letters and N be the set of integers from 1 to 26. Let $f : L \to N$ be a function corresponding to a particular anagram of the alphabet such that $f(l) = i$ means that the letter l is the ith character in the anagram. For example for f corresponding to the regular alphabet ordering $f(A) = 1; f(D) = 4$. How many anagrams of the alphabet:

   - Satisfy $f(A) < f(B)$(A comes before B)
   - Satisfy $f(A) < f(B)$ AND $f(C) < f(D)$
   - Satisfy $f(A) < f(B)$ OR $f(C) < f(D)$
   - Satisfy $f(A) < f(B) < f(C) < f(D)$
   - Satisfy $f(A) < f(B)$ AND $f(C) < f(D)$ AND $f(C) < f(B)$

3. (Nicholas) This problem is an exploration into the Hockey Stick Identity. Consider the set $S = \{1, 2, 3, \dots, n\}$. We will find the number of subsets of S that are of size k in two ways.

   (a) Calculate the number of ways to choose k numbers from n total numbers.

   (b) Sum up the total number of subsets of size k that contain 1 as the smallest integer, 2 as the smallest integer, ..., up to $n-k+1$ as the smallest integer (why do we stop at $n-k+1$?). *Note: this is somewhat similar to Homework 2 Problem 6*

   Reason to yourself why these two expressions should be equivalent.

   - The Hockey Stick Identity is the relation between the binomial coefficients calculated in parts (a) and (b). What does the Hockey Stick Identity state?
   - Set n to be 7 and k to be 3 (or choose any valid n and k you want). Redo this problem using those values. Locate all of the binomial coefficients calculated in parts (a) and (b) in Pascal's Triangle. Do you see why it's called the Hockey Stick Identity?

4. (Saad) Use the Binomial Theorem on $(1+2)^n$ to get an interesting identity.

5. (John) There is a yearly tournament consisting of 10 wizards to determine the most powerful mage in the continent, where each duel will be a one versus one between 2 wizards. What is the total number of ways we could arrange duels for every wizard in the initial bracket of the tournament (so 5 duels between 10 wizards)? Now, the wizard council overseeing the tournament wants to begin each duel with an anagram of "BLESSED BY MANA." How many distinct ways can the council name the duels?

6. (Randy) Suppose that we have functions $f : X \to Y$ and $g : Y \to Z$, if $(g \circ f)(x) = g(f(x))$ is a bijection from X to Z, what can we say about f? What can we say about g?

7. (Zach) Suppose we are creating a password for the web login of our bank account. Since this is where our money is stored, the bank's website has strict rules for creating a password to prevent unauthorized access. We are allowed to use uppercase (A-Z) and lowercase characters (a-z), and the following symbols: (!,@,-,*). Our password must

be a minimum of 5 characters and a maximum of 7 characters. The password must start and end with a symbol and must include a symbol somewhere in between. The other characters besides the symbol character somewhere in the middle and on the ends must be upper- or lower- case characters. Repeats of any character are allowed. How many possibilities are there for our new password?

8. (Matthew) This is a expansion of my problem from last week. A small city has 3 neighborhoods. Let n_i denote the ith neighborhood. We want to house 70 guests and the following is true:

   - n_1 has 3 homes and will house 16 people
   - n_2 has 7 homes and will house 43 people
   - n_3 has 2 homes and will house 11 people

   How many ways are there to house all guests if:

   (a) We only care about the headcount of each house

   (b) We care only about the headcount of each house and who goes in which neighborhood

   (c) We care about who goes in which neighborhood and which house they are in

Solutions

1.
   - Consider the following function:

$$f : S \to T$$

$$f(i,j) = (i, j-i)$$

This is a valid function since every element of S maps to some element in T: both $x = i$ and $y = j - i$ are elements of $\mathbb{N}$ (because $j > j$), and $x + y = i + (j - i) = j \leq n$.

To show that f is one-to-one, say that $f(i,j) = f(i',j')$. This means $(i, j-i) = (i', j'-i')$. Therefore:

$$i = i'$$

$$j - i = j' - i'$$

This means $i = i'$ and $j = j'$, so $(i,j) = (i',j')$. Done.

To show f is onto, consider any $(x,y) \in T$. Let $(i,j) = (x, x+y)$. Observe that $f(i,j) = f(x, x+y) = (x, x+y-x) = (x,y)$. Now, since $(x,y) \in T$, we have $x, y \in \mathbb{N}$ and $x + y \leq n$. This implies $x < x + y \leq n$, which means $i < j \leq n$ as desired, so $(i,j) \in S$. Done.

   - It is clear that S represents all ordered pairs (i,j) that we can make from $\{1, 2, \ldots, n\}$, such that $i < j$. This is half the number of ordered pairs, which is $n(n-1)$. So $|S| = n(n-1)/2 = \binom{n}{2}$. So $|T| = \binom{n}{2}$ as well because of the bijection.

   - We need $x + y + z = n$ with $x \geq 1$, $y \geq 1$, and $z \geq 0$. We can rewrite this (as we have seen in class) as:

$$x' + y' + z = n - 2$$

where $x', y', z \geq 0$. The number of solutions is $\left(\!\binom{3}{n-2}\!\right) = \binom{3+n-2-1}{3-1} = \binom{n}{2}$, as before.

   - There are $n-1$ possible pairs for $x + y = n$, these are $(1, n-1), (2, n-2), \ldots (n-1, 1)$. Similarly, there are $n-2$ possibilities for $x+y = n-1$, and so on, until we reach 1 possibility for $x+y = 2$. By the addition rule, we have $(n-1) + (n-2) + \ldots + 1 = \binom{n}{2}$ possible pairs.

2.
   - Let J be the set of all anagrams where the letter A comes before B. Let K be the set of all anagrams where B comes before A. Define $g : J \to K$ as a function that swaps A and B in the anagram, so g maps all of the anagrams where A comes before B to anagrams where B comes before A. Since $g(g(j)) = j$, we can conclude that $g^{-1}(j) = g(j)$. Since g has an inverse it is a bijection (that's another way to show bijection) and therefore $|J| = |K|$ and since A and B have to either be in order or in reverse order $|J| = \#$ of anagrams / 2 = 26!/2
   - You can follow an argument similar to the previous part with a function that swaps letters C and D acting on sets P(the set of all anagrams where A comes before B AND C comes before D) and Q(the set of all anagrams where A comes before B AND D comes before C). You will arrive at the conclusion that $|P| = |Q|$ and since $|P| + |Q| = |J|$ then $|P| = 26!/4$
   - Let S be the set of all anagrams where the letter C comes before D. It's apparent that $|S| = |J|$ and by inclusion-exclusion principle the answer is: $|S| + |J| - |P| = (3/4) * 26!$
   - Choose 4 positions for A, B, c, D in $\binom{26}{4}$ ways (and place them in order). Then permute the rest of the 22 letters in 22! ways. We get by the product rule $\binom{26}{4}22! = 26!/4!$.
   - Same as above, except that the four letters can be placed in one of 5 orders: (A, C, D, B), (A, C, B, D), (C, D, A, B), (C, A, D, B), (C, A, B, D), giving $5 \cdot 26!/4!$.

3. (a) $\binom{n}{k}$

   (b) The number of subsets of size k that contain 1 as the smallest integer is the same as the number of ways to choose $k-1$ numbers from $n-1$ options (1 is already included in the subset). This is equal to $\binom{n-1}{k-1}$.
   Next, the number of subsets of size k that contain 2 as the smallest integer is the same as the number of ways to choose $k-1$ numbers from $n-2$ options (can't use 1, and 2 is already included). This is equal to $\binom{n-2}{k-1}$.
   We continue this until $n-k+1$ is the smallest integer in the subset (stop at $n-k+1$ because this is the largest integer where with n elements, we can still form a valid size k subset). This is the same as the number of ways to choose $k-1$ numbers from $n-(n-k+1) = k-1$ options. This is equal to $\binom{k-1}{k-1}$.

Summing up all of these subsets gives us:

$$\binom{n-1}{k-1} + \binom{n-2}{k-1} + ... + \binom{k-1}{k-1}$$

While (a) is a more direct calculation for the solution of this problem, (b) uses the idea of disjoint sets and the addition rule to solve it in another way. Hence, these expressions are equivalent.
Putting these ideas together, the Hockey Stick Identity states:

$$\sum_{i=k-1}^{n-1} \binom{i}{k-1} = \binom{n}{k}$$

or more commonly expressed as:

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Using the example of n = 7 and k = 3, the solutions that you would get are:

$$\binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \binom{7}{3}$$

or equivalently:

$$15 + 10 + 6 + 3 + 1 = 35$$

Locating these values in Pascal's Triangle...

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & \boxed{1} & & & & & \\
 & & & & 1 & & 3 & & \boxed{3} & & 1 & & & & \\
 & & & 1 & & 4 & & \boxed{6} & & 4 & & 1 & & & \\
 & & 1 & & 5 & & \boxed{10} & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & \boxed{15} & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & \boxed{35} & & 35 & & 21 & & 7 & & 1
\end{array}$$

They're in the shape of a hockey stick!

4.
$$(1+2)^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \ldots + \binom{n}{n}2^n = 3^n$$
So
$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$
and in general:
$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

5. Since each duel is a one versus one consisting of two mages, the number of combinations for each duel is equal to $\binom{n}{2}$, where n is the number of remaining wizards. The first duel has $\binom{10}{2}$ ways of choosing a pair of wizards to duel, the second has $\binom{8}{2}$, and this continues with n decrementing by 2 each duel until we reach the last one, which is $\binom{2}{2}$. The total number of possible combinations would then be $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 113400$. However, we must also take into account the over-counting occurring for each selection of pairs, which is equal to 5!. Therefore, the total number of combinations to arrange the initial 5 duels of the tournament is $113400/5! = 945$.
The second part of the problem is an anagrams question. There are a total of 15 characters in the phrase "BLESSED BY MANA," with the following characters repeating: $B = 2\,times$, $E = 2\,times$, $S = 2\,times$, $'\ ' = 2\,times$, $A = 2\,times$. Therefore the total number of anagrams is $15!/2!2!2!2!2! = 40,864,824,000$.

6. We know that f is an injection, because if it weren't then we would have $f(x) = f(y)$ for $x, y \in X$ but $x \neq y$ thus $(g \circ f)(x) = (g \circ f)(y)$ contradicting the fact that $g \circ f$ is a bijection. Likewise g is a surjection because if it were not then there would be a $z \in Z$ such that no $y \in Y$ would satisfy $z = g(y)$ contradicting the fact that $(g \circ f)$ is a bijection.

7. Since the number of possibilities for each length are disjoint, we figure out how many passwords there are for each length, and add the result. The sum is our answer. For each length, we know that the password must start and end with a symbol, which means multiplying by 4*4 = 16. It doesn't matter when we multiply, because multiplication is commutative. Let's figure out how many possibilities there are for whatever comes in between, and multiply the result by 16 to get the

number of possibilities for each length. For what comes in between, there are 52 possibilities for each character (26 upper case chars + 26 lower case) and we are allowed repeats, but there is a catch. One of the characters must be a symbol. The rest are upper- and lower-case characters. So, if for instance, we have a password of length 5, we have 3 chars in between, and 2 of them must be upper- and lower-case chars, so it's 52*52. There are 4 possibilities for the 3rd char (the symbol), so we multiply by 4. So we have 52*52*4. This doesn't account for the fact that the symbol char somewhere in the middle can be in any position, not just in one place. So we multiply by the number of positions the symbol char can be in. So we have 52*52*4*3 Next, we multiply by 16 to account for the symbol chars on the ends. So, we have (52*52*4*3)16 possibilities for a password of length 5.

Next, we repeat this entire process for a password of length 6 and 7. We end up with (52*52*52*4*4)16 possibilities for a password of length 6 and (52*52*52*52*4*5)16 possibilities for a password of length 7.

Finally we add the results. So, we have (52*52*4*3)16 + (52*52*52*4*4)16 + (52*52*52*52*4*5)16 possibilities for our password of length 5-7 chars.

8. (a) If we only care about the headcount of each house, then we can simply multiply the amount of ways to house 16 guests in 3 homes, 43 guests in 7 homes, and 11 guests in 2 homes. To do this, we have to find the total amount of solutions for the following equations:
   $x_1+x_2+x_3=16$
   $x_1+x_2+x_3+x_4+x_5+x_6+x_7=43$
   $x_1+x_2=11$
   To find the total amount of solutions for each equation, we can employ stars and bars. We will get that the amount of ways to house the respective amount of guests in each neighborhood is $\binom{18}{2}$, $\binom{49}{6}$, and $\binom{12}{1}$. Therefore, our result for this part is $\binom{18}{2}\binom{49}{6}\binom{12}{1}$.

   (b) If we now care about who goes in which neighborhood along with the headcount of each house, we have to count the total amount of ways to assign the 70 people a neighborhood, then multiply the total amount of ways to house the guests if we only care about the headcount (previous part).

- Assigning 16 people to n_1: $\binom{70}{16}$
- Assigning 43 people to n_2: $\binom{54}{43}$
- Assigning 11 people to n_3: $\binom{11}{11} = 1$

Now that we've found the total amount of ways to assign the 70 people a neighborhood, we can then multiply the answer we got from the previous part by this number. Our result is $\binom{70}{16}\binom{18}{2}\binom{54}{43}\binom{49}{6}\binom{12}{1}$.

(c) If we now start thinking about the individual, we have to account for more than just the headcount of each home. In other words, if we have a house that can hold two people, having Alice and Bob in there is different from having Bob and Carol in there, despite housing 2 people both times. To calculate the total amount of ways to house all guests given the conditions that we care about the individual and which house they are in, we have to first find out the total amount of ways to assign people a home. Now that we're considering the individual, each person is making a choice, so we will need to multiply the total amount of ways a person can choose a home by the previous amount.

- $\binom{70}{16}$ ways to assign people to n_1. Each person has 3 choices (homes to choose), so multiply by 3^{16} since there is 16 people
- $\binom{54}{43}$ ways to assign people to n_2. Each person has 7 choices, so multiply by 7^{43}
- $\binom{11}{11}$ ways to assign people to n_3. Each person has 2 choices, so multiply by 2^{11}

Our final answer is $\binom{70}{16}(3^{16})\binom{54}{43}(7^{43})(2^{11})$.

Week 5

Introduction to logic, propositions, logical operators: AND, OR, NOT, Implication, Iff. Truth tables and Boolean functions. What makes $P \Rightarrow Q$ true, $\forall x, P(x)$ true, and $\exists x, P(x)$ true.

1. (Saad) Why is the following true?

$$\text{Pigs can fly} \Rightarrow 15 \text{ is prime}$$

2. (Saad) Prove the following is false:

$$\forall x \in \mathbb{N}, x \text{ is prime} \Rightarrow x \text{ is odd}$$

   in two ways:

   - by providing a counter example
   - by negating it and providing an example (thus showing that the negation is true). Observe that the negation of $\forall x, P(x)$ is $\exists x, \neg P(x)$. Here, $P(x)$ is: x is prime $\Rightarrow x$ is odd. So recall how to negate something of the form $P \Rightarrow Q$.

3. (Randy) We would like to find the sum of two bits a and b. Let s denote their sum and c the carry. Can you write boolean expressions for the values of s and c?

4. (Randy) Suppose that we want to know $a_1 \wedge a_2 \wedge ... \wedge a_n$ where the a_is for $1 \leq i \leq n$ are propositions. Suppose that we know the value for a single a_i, how many of the propositions do we need to check to determine the value of the large expression? Suppose now we are interested in the value of $a_1 \vee a_2 \vee ... \vee a_n$ and we again know a_i, how many of the propositions do we need to check to determine the value of the large expression?

5. (Zhen) Given $P \Rightarrow Q$ evaluates to 0, what does the following expression evaluates to?

$$\neg P \vee \neg Q$$

6. (Zhen) You encountered a shady man who promised that if you give him 5 Dogecoins, he'll give you a guinea pig.

   (a) In what scenario can you claim that his promise is false?

(b) If you don't give him the 5 Dogecoins, can you claim that his promise is false? If not, what's the only other option? (Now can you see why the truth table for $P \Rightarrow Q$ is the way it is?)

7. (Vlad) Let a, b, c be boolean variables. Describe a procedure to construct a boolean expression that is only False when a, b and c take on a specific configuration. For example how do you construct an expression that is only false when a = 1, b = 0, c = 1?

8. (John) Three gatekeepers guard a magic tower, and each one only tells the truth or always lies. The following propositions are told:

   (a) Gatekeeper A says, "If I am telling the truth, then Gatekeeper B is lying."

   (b) Gatekeeper B says, "Gatekeeper A and C are either both lying or both telling the truth."

   (c) Gatekeeper C says, "Gatekeeper A is telling the truth."

   Which gatekeepers are lying and which ones are telling the truth?

9. (Tasmina) Suppose you encounter a magical well which will give you an infinitely large fortune, provided that you answer its riddle.

   (a) Write this statement in terms of a logical proposition (i.e. A implicates B).

   (b) The riddle states, "I am either male or female. If you can prove the logical value of 'I am male $\Rightarrow$ You are human' then you will pass." What is the answer to the riddle?

10. (Tasmina) You and your friends are trying to figure out dinner reservations. You have to accommodate 4 diets: vegan, pescatarian, gluten-free, kosher. You must also find places that are safe from peanut allergens. After much discussion, the finalized suggestions for cuisine are Italian, Korean, and Moroccan. Write all these limitations using logical operators (i.e. AND, OR, NOT). Note, you many not need them all.

11. (Zach) I know that when I arrive home, my dog wags his tail. So, what conclusion can I make about the truthfulness of this statement if I take my dog for a walk? If I give him a treat? What about if I take my dog for a walk and give him a treat during the walk, and then arrive home and my dog doesn't wag his tail? Justify your answers.

Solutions

1. Pigs can't fly. The statement is therefore of the form $P \Rightarrow Q$, where P is false. This makes the statement true.

2. For the statement to be true, every integer in $\mathbb{N}$ must make this true:

$$x \text{ is prime} \Rightarrow x \text{ is odd}$$

   Here's a counter example when $x = 2$: (2 is prime $\Rightarrow$ 2 is odd) has the form $True \Rightarrow False$, which is false.

   If we negate the statement, we get:

$$\exists x \in \mathbb{N}, x \text{ is prime} \wedge \neg(x \text{ is odd})$$

   here we change $\forall$ to $\exists$, and we negate $P \Rightarrow Q$ by using $P \wedge \neg Q$. Obviously, $x = 2$ is an example that makes this statement true: It is prime, and it is not odd.

3. One solution is $s = a \otimes b = (\neg a \wedge b) \vee (a \wedge \neg b)$ and $c = a \wedge b$. We call the operation $\otimes$ xor, exclusive or, which is true only when a and b are distinct.

4. For the former, if a_i is false then we do not need to check the other propositions (why?), otherwise we will need to keep checking until we find a false proposition. For the latter, if a_i is true then we do not need to check anymore (why?), otherwise we will need to keep checking until we find a true proposition.

5. Given $P \Rightarrow Q$ evaluates to 0, that means P is 1 and Q is 0. Therefore, we have $0 \vee 1$, which is 1.

6. A scenario would be to give him the 5 Dogecoin and in return he doesn't give you the guinea pig. If you don't give him the 5 Dogecoins, then you can't claim that his promise is false, thus making his promise true by default.

7. We know that $d \vee e \vee f$ is only false when $d = 0, e = 0, f = 0$. We can replace d, e, f with expressions containing a, b, c such that when a, b, c are in the configuration we want to give false d, e and f all become 0. For example if we want the expression to be false when $a = 1, b = 0, c = 1$ we can let $d = \neg a$, $e = b$, $f = \neg c$. So in this case our expression will be $\neg a \vee b \vee \neg c$

8. We can represent Gatekeeper A's statement as such: $T_A \Rightarrow \neg T_B$. This means that if Gatekeeper A is telling the truth, Gatekeeper B *must* be lying. If Gatekeeper A is lying, then we cannot make any conclusive claims. Let us analyze both cases:

   (a) Gatekeeper A is telling the truth (T_A is true $\Rightarrow$ T_B is false): This means that Gatekeeper B is lying. If B is lying, then for their statement to be false, C must also be lying (since B's claim that $T_A = T_C$ is false when $T_A \neq T_C$ is true. We know that T_A is true, so T_C must be false). If C is lying, then $\neg T_A$ is true, meaning that Gatekeeper A is lying. However, this is a contradiction, as Gatekeeper A was said to be telling the truth. Therefore, this combination consisting of a truthful Gatekeeper A does not work.

   (b) Gatekeeper A is lying: This means that $T_A \Rightarrow \neg T_B$ is true regardless of T_B's value because a false premise makes the implication true. We then take into consideration the two values of T_B. If T_B is true, then Gatekeeper C must be lying (since Gatekeeper A is lying). If Gatekeeper C is lying, then that means Gatekeeper A is lying since $\neg T_A$ evaluates to Gatekeeper A is lying. Gatekeeper A is indeed lying, making this combination to be true. If T_B is false, then Gatekeeper C must be telling the truth. This means that T_A is true, and Gatekeeper A is telling the truth. However, since we established that Gatekeeper A is lying, this is a contradiction. Therefore, the only correct answer is that Gatekeeper A is lying, Gatekeeper B is telling the truth, and Gatekeeper C is also lying.

9. (a) This can be written as: The magical well will give you an infinitely large fortune IFF you answer its riddle.

   (b) You don't know the gender of the well, but you do know that you are human. If you look at the truth table, you'll see that $0 \Rightarrow 1$ and $1 \Rightarrow 1$ are both true. Thus, this statement will always be true.

10. This can be written as: (Vegan AND Pescatarian AND Gluten-free AND Kosher) AND (NOT (Peanuts)) AND (Italian OR Korean OR Moroccan)

11. I can conclude that the statement is still true if I take my dog for a walk, and in the case that I give my dog a treat. In the case where I take my dog for a walk and give him a treat during the walk and

then arrive home but he doesn't wag his tail, I can conclude that the statement is false. This is because the statement doesn't mention what happens when I take my dog for a walk or give him a treat, it only mentions what happens when I arrive home, so taking him for a walk or giving him a treat doesn't disprove the statement, regardless of whether or not my dog wags his tail. If I take him for a walk and give him a treat during the walk and then arrive home, I am still arriving home, so if my dog doesn't wag his tail it means the implication is false.

Week 6

Proofs: direct, contradiction, contrapositive, existential, case analysis, parity, etc...

1. (Saad) Prove by contradiction that a triangular number cannot be a power of 2. *Hint*: if $ab = 2^x$ then both a and b are powers of 2.

2. (Saad) Consider the set $S = \{1, 2, \ldots, 100\}$. Prove by contradiction that there is no $x \in S$ such that $2x$ is equal to the sum of all other integers in S.

3. (Saad) Use the contrapositive to prove the following, where x and y are integers:
$$x + y \text{ is odd} \Rightarrow x \neq y$$

4. (Saad) Prove by case analysis (try all) that if n is not a multiple of 3, then n^2 is not a multiple of 3. *Hint*: consider all cases of the remainder in the division of n by 3. Can you state the contrapositive?

5. (Saad) Prove that there exists a Fibonacci number greater than 1 that is a square.

6. (Vlad) Prove that for $n >= 5$ if n is prime then $n^2 - 1$ is a multiple of 24. Hint: You will need to use the difference of squares formula.

7. (John) We are given three distinct numbers a, b, and c, such that the sum of any two of these numbers will give a result always divisible by 2. Prove that all three numbers must have the same parity.

8. (Tasmina) Prove that if the integer product ab is even, then at least one of a or b is even.

9. (Nicholas) Imagine that you are coloring numbers on a number line (starting at 1) red or blue. You want to color the numbers such that it is impossible to use any two numbers of the same color (not necessarily distinct) to sum up to another number of that color. For example, if you choose to color "1" red, you can not color "2" red because otherwise $1 + 1 = 2$. Prove by contradiction that the largest number you can color up to is "4".

10. (Nicholas) Prove that there exists a sequence of 2024 consecutive composite numbers. *Hint*: Similar to Homework 6 Question 1, the key to

showing that a number isn't prime is by showing that you're able to factor out a number greater than 1 from it. Example: $4! + 4$ is a composite number because we can factor out 4 from it.

Does there exist a sequence of n consecutive composite numbers? If so, what number should we start with?

11. (Zach) Prove that there exists a perfect square such that its square is equal to 41476 divided by 2 minus 2. What is the principal square root of this perfect square?

Solultions

1. Assume that there is a triangular number that is a power of 2; for instance, $T_n = 2^k$, then:

$$T_n = \frac{n(n+1)}{2} = 2^k \Rightarrow n(n+1) = 2^{k+1}$$

   This means both n and $n+1$ are powers of 2, and they are both even. But one of them must be odd, a contradiction.

2. Let's assume that there is $x \in S$, such that $2x$ is equal to the sum of all other integers in S. This means that the sum of all integers in S must be equal to $3x$ and, therefore, $100 \times 101/2 = 3x$. So $5050 = 3x$ and $x = 5050/3$, a contradiction since 5050 is not divisible by 3.

3. The contrapositive is:

$$x = y \Rightarrow x + y \text{ is even}$$

   Here's a proof:

$$x = y \Rightarrow x + y = x + x = 2x \Rightarrow x + y \text{ is even.}$$

4. There are only two cases to consider for n not a multiple of 3:

   - Remainder of 1: $n = 3k+1 \Rightarrow n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3k' + 1$
   - Remainder of 2: $n = 3k+2 \Rightarrow n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1 = 3k' + 1$

   The contrapositive: n^2 is a multiple of 3 $\Rightarrow$ n is a multiple of 3. Observe that we can strengthen this to iff:

$$n \text{ is a multiple of } 3 \Leftrightarrow n^2 \text{ is a multiple of } 3$$

   This is because if n is a multiple of 3, then n^2 is also a multiple of 3 ($n = 3k \Rightarrow n^2 = 9k^2 = 3(3k^2)$).

5. Here's the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

   Observe that 144 is 12^2.

6. We can rewrite $n^2 - 1$ as $n^2 - 1^2$ and use the difference of squares formula to obtain $n^2 - 1^2 = (n-1)(n+1)$. Since we know that n is prime and that $n >= 5$ we can conclude that n is odd. Since n is odd we can conclude that $n-1$ and $n+1$ are even, furthermore either $n-1$ or $n+1$ must be a multiple of 4 since every 2nd even number is a multiple of 4 and we have two "consecutive" even numbers $n-1$ and $n+1$. Every third number is a multiple of 3. n can't be a multiple of 3, since it is prime. Therefore either $n-1$ or $n+1$ is a multiple of 3. Therefore multiplying $(n-1)*(n+1)$ will always result in multiplying a multiple of 4 a multiple of 2 that is distinct from a multiple of 4 and a multiple 3 resulting in a multiple of $2*4*3 = 24$

7. We will prove by contradiction. Assume the contrary and let us try to prove that the original statement is true when not all integers a, b, and c share the same parity. Without loss of generality, let us say that a is even, b is odd, and c is even. This means $a = 2k$ and $b = 2l+1$ where $k, l \in \mathbb{Z}$. The sum of a and b gives us $2k+(2l+1) = 2(k+l)+1$, reaching a contradiction since $2 \nmid 2(k+l)+1$.

8. The contrapositive is

$$\text{a and b are both odd} \Rightarrow \text{ab is odd}$$

Here's a proof:

$$\begin{gathered}
\text{a and b are both odd} \\
\Rightarrow a = 2x+1 \text{ and } b = 2y+1 \\
\Rightarrow ab = (2x+1)(2y+1) \\
\Rightarrow ab = 4xy + 2x + 2y + 1 \\
\Rightarrow ab = 2(2xy + x + y) + 1 \\
\Rightarrow ab = 2k+1 \\
\Rightarrow ab \text{ is odd.}
\end{gathered}$$

Since we have proved the contrapositive, we know the original statement is true.

9. Assume that we are able to color our numbers up to a number greater than "4". Without loss of generality, color the number "1" red. Since "1" is red, "2" must be blue because otherwise we would have $1+1 = 2$. There are no restrictions for "3", so for now, we will skip this number.

"4" must be red because otherwise $2+2=4$. Now that "4" is red, "3" must be blue because otherwise $1+3=4$. "5" must be red because otherwise $2+3=5$, but "5" must also be blue because otherwise $1+4=5$. "5" can not be colored both red and blue, therefore, we have a contradiction.

If you're interested in a problem like this, try to find the largest number you can reach using three colors.

If you're even more interested, look into Schur numbers and this video (https://www.youtube.com/watch?v=nfynJIb5tyg). It's a neat video on how propositional logic can be used to solve extremely computationally intensive tasks such as finding the largest number you can reach using five colors.

10. Imagine what our sequence of 2024 numbers would look like: n, $n+1$, ..., $n+2022$, $n+2023$. To make sure that all of them are composite, we want to guarantee that we're able to factor out a number greater than 1 from them. Starting from $n+2023$, we can imagine factoring out 2023 and getting $n+2023 = 2023(n/2023+1)$. For $n+2022$, factor out 2022 and get $n+2022 = 2022(n/2022+1)$. We can continue doing this until $n+2$ but can't do it for $n+1$ or n because otherwise would get $n+1 = 1(n/1+1)$ and $n = n(1)$, and there's no guarantee that those numbers aren't prime. The next question we have to ask is how do we make sure that all of our n/k will be integer values for $k \in [2, 2023]$. We make $n = 2023!$. But actually we want $n = (2023+2)!$ because then we can apply this factoring out trick for numbers $n+2$ to $n+2025$ and get a guaranteed sequence of 2024 consecutive composite numbers, rather than 2022. Therefore, if we set our first number in the sequence to be equal to $2025!+2$ (so our last number is $2025!+2025$), we can show that we will be guaranteed a sequence of 2024 consecutive composite numbers.

    Using the same idea, we can show that there does exist a sequence of n consecutive composite numbers. If we start our sequence at $(n+1)!+2$ and therefore end our sequence at $(n+1)!+(2+n-1) = (n+1)!+(n+1)$, we can factor out every number from 2 to $(n+1)$ and get a sequence of n consecutive composite numbers.

11. The perfect square is 144, and its principal square root is 12.

Week 7

Infinity, countable and uncountable sets, inclusion-exclusion, pigeonhole.

1. (Saad) Find a bijection between $\{x \in \mathbb{R} | 0 < x < 1\}$ and $\mathbb{R}$. What can we conclude?

2. (Saad) The set of all infinite binary sequences in uncountable. What if we consider the set of all infinite binary sequences that do not contain consecutive 1s. Use the diagonalization method to prove that this set is still uncountable. *Hint*: think about each infinite sequence as a sequence of pairs of bits; for instance, 10010100 . . . is 10 01 01 00 . . .

3. (Saad) Assume $n = p_1p_2p_3$, where p_1, p_2, and p_3 are prime. How many integers less or equal to n are not divisible by p_1 and not divisible by p_2 and not divisible by p_3? Can you show that this number is
$$(p_1 - 1)(p_2 - 1)(p_3 - 1)$$

4. (Saad) Consider a finite length segment AB. Now consider the set of all points on that segment. Is that set of points countable?

5. (Saad) Given $n + 1$ integers, prove that 2 of them, say x and y, must satisfy $x - y$ is a multiple of n. *Hint*: in the division by n, how many possible remainders are there?

6. (Tasmina) You are with 12 of your friends. You are all going to sit around a circular table with 24 seats. Prove that at someone will be sitting next to someone.

7. (John) Consider the set $\mathbb{N}$ (all natural numbers) and let $f : \mathbb{N} \to \{1, 2, 3, ..., 100\}$. Prove that there exists infinitely many numbers $n_1 < n_2 < n_3 < ...$ such that $f(n_1) = f(n_2) = f(n_3) = ...$
Hint: Which discrete principle do we need to prove this?

8. (Randy) Is $\mathbb{N} \times \mathbb{N} = \{(x, y) : \forall x, y \in \mathbb{N}\}$ countable or uncountable?

9. (Zhen) In a cooking contest among 300 people, each person has to pick 5 out of the 10 available ingredients to make their winning dish. Show that at least 2 people will choose the same set of ingredients.

10. (Zach) You don't have a lot of time on your hands so you outsource your laundry. Unfortunately, you forgot how many of each color socks

you own, and your laundry was just returned to you so you have to count them. You know that you have 50 socks in total, and that they are either red, blue or both. You also know that you have twice as many red socks as blue socks, and that 10 of your socks are both red and blue. How many red socks do you own? How many blue?

11. (Nicholas) Consider a group of 5 people of varying heights. In how many ways can you order them so that no group of 3 consecutive people are ever standing in increasing order of height?

Solutions

1. Consider the function $f(x) = \ln(\frac{x}{1-x})$. This maps the interval $(0,1)$ to $\mathbb{R}$. It can be shown that this is a bijection. So $(0,1)$ and $\mathbb{R}$ have the same "size".

   one-to-one:

   $$f(x) = f(x') \Rightarrow \frac{x}{1-x} = \frac{x'}{1-x'} \Rightarrow x(1-x') = x'(1-x)$$
   $$\Rightarrow x - xx' = x' - x'x \Rightarrow x = x'$$

   onto: Let $y \in R$, and consider x such that $f(x) = y$. Then

   $$\ln(\frac{x}{1-x}) = y \Rightarrow \frac{x}{1-x} = e^y \Rightarrow x = (1-x)e^y$$
   $$\Rightarrow x(1+e^y) = e^y \Rightarrow x = \frac{e^y}{1+e^y}$$

   and observe that $0 < x < 1$.

2. We can use Cantor's diagonalization proof, and create an infinite binary sequence s such that no $i \in \mathbb{N}$ satisfies $f(i) = s$. We make the i^{th} pair of bits in s different from the i^{th} pair of bits in $f(i)$, using the following modification mechanism: $01 \to 00$, $10 \to 00$, and $00 \to 01$. This way, all 1s in s are in even positions, and therefore s has no consecutive 1s. Done.

3. First, we find how many integers are divisible by p_1 or p_2 or p_3. By inclusion-exclusion, this number is:

   $$\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} - \frac{n}{p_1p_2} - \frac{n}{p_1p_3} - \frac{n}{p_2p_3} + \frac{n}{p_1p_2p_3}$$

   Then we subtract that number from n, to get:

   $$n - \frac{n}{p_1} - \frac{n}{p_2} - \frac{n}{p_3} + \frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \frac{n}{p_2p_3} - \frac{n}{p_1p_2p_3}$$

   Replacing n by $p_1p_2p_3$, we get:

   $$p_1p_2p_3 - p_2p_3 - p_1p_3 - p_1p_2 + p_3 + p_2 + p_1 - 1$$

   which is exactly (try)

   $$(p_1 - 1)(p_2 - 1)(p_3 - 1)$$

4. The set of points on a finite length segment is uncountable. We can make a bijection from that set of points to $[0, 1]$ (and we know already that the interval $(0, 1)$ in uncountable). Given a point P on the segment AB, $f(P)$ will be the length AP divided by the length of AB. It is easy to see that this is a bijection.

5. By pigeonhole, two of the integers will have the same remainder in the division by n, since we have $n+1$ integers and only n remainders. Therefore $x = kn + r$ and $y = k'n + r$. Finally, $x - y = (k - k')n$, which is a multiple of n.

6. You must remember to include yourself with your 12 friends, to make a total of 13 people. Since there are only 24 seats (12 pairs) then, by pigeonhole principle, at least two people must be neighbors (two must belong to the same pair).

7. We want to prove that infinitely many values within the set of natural numbers map onto the same value (defined by function f). To accomplish this, we will apply the pigeonhole principle. Here, the 'pigeons' are the infinitely many natural numbers, while the 'holes' are the 100 values f can take. The pigeonhole principle claims that if an infinite number of pigeons are distributed into a finite number of holes, then at least one hole $k \in \{1, 2, 3, ..., 100\}$ must contain an infinite number of pigeons (sometimes referred to as the infinite pigeonhole principle). To see this, consider the set $S = \{1, 2, \ldots, n\}$. There is a $k \in \{1, 2, \ldots, 100\}$ such that at least $\lceil n/100 \rceil$ elements in S map to k by pigeonhole. As n goes to infinity, this number keeps growing.

   Another way is a proof by contradiction. Assume that every $k \in \{1, 2, \ldots, 100\}$ has a finite number of integers in $\mathbb{N}$ that map to it. This means that the number of integers in $\mathbb{N}$ is finite, a contradiction. Therefore, some k must have infinitely many integers in $\mathbb{N}$ that map to it.

8. If we group elements $(x, y) \in \mathbb{N} \times \mathbb{N}$ by their sum $x + y$ then each set $\{(x, y) : x+y = n\}$ for fixed n is finite. We can count these pairs starting from $n = 0$ like so $(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (0, 3), ...$ This way we can always associate an arbitrary (x, y) with an appropriate number.

9. There are 252 ways to choose 5 ingredients from 10 (10 choose 5). Using pigeonhole, the ceiling of 300/252 will give 2.

10. You own 40 red socks and 20 blue socks. You need to solve the system of equations r + b - z = 50, and r = 2b, where r is the number of red socks, b is the number of blue socks, and z is the number of socks that are both red and blue. z = 10 is given.

11. We will use the inclusion-exclusion principle to solve this question. There are three possible positions for where this group of three people can stand. Let A be the event where the first, second, and third people are standing in ordered height. Let B be the event where the second, third, and fourth people are standing in ordered height. And let C be the event where the third, fourth, and fifth people are standing in ordered height.

    Our final answer will be equal to:
    $5! - (|A| + |B| + |C| - |A \cup B| - |A \cup C| - |B \cup C| + |A \cup B \cup C|)$

    $|A|$, $|B|$, and $|C|$ can all be determined by:
    1. choosing the three people who will be in ordered height ... $\binom{5}{3}$ ways
    2. placing the remaining people in the two remaining positions: 2! ways
    This gives you $|A| = |B| = |C| = 20$.

    $|A \cup B|$ and $|B \cup C|$ can be determined by:
    1. choosing the four people who will be in ordered height ... $\binom{5}{4}$ ways
    2. placing the remaining person in the remaining position: 1 way
    This gives you $|A \cup B| = |B \cup C| = 5$

    $|A \cup C|$ can be determined by:
    1. placing all people in ordered height ... 1 way
    This gives you $|A \cup C| = 1$

    Lastly, $|A \cup B \cup C|$ can also be determined by:
    1. placing all people in ordered height ... 1 way
    This gives you $|A \cup B \cup C| = 1$

    Plugging in our values into our final answer, we get:
    $120 - (20 + 20 + 20 - 5 - 5 - 1 + 1) = 70$ valid arrangements

Week 8

Proofs by induction.

1. (Vlad) A flower figure consists of stars(*) and zeros(0). If a flower figure has a size n then it can be embedded in a 2n+1 by 2n+1 grid. The following ascii art figures are meant to represent flowers of sizes 1, 2 and 3 respectively:
   size 1:
   0 * 0
   * 0 *
   0 * 0
   size 2:
   0 0 * 0 0
   0 * * * 0
   * * 0 * *
   0 * * * 0
   0 0 * 0 0
   size 3:
   0 0 0 * 0 0 0
   0 0 * * * 0 0
   0 * * * * * 0
   * * * 0 * * *
   0 * * * * * 0
   0 0 * * * 0 0
   0 0 0 * 0 0 0

   Figure out the pattern that these shapes follow and prove by induction that a flower of size n will have 4n + 4n(n-1)/2 = 4n(n+1)/2 = 2n(n+1) stars.

2. (Tasmina) Prove using induction that the total number of distinct ways to carve faces on n pumpkins, where each pumpkin can be either "happy" or "scary," is 2^n.

3. (Saad) We are given that $x + 1/x$ is an integer. Prove that for all $n \geq 0$,
$$x^n + 1/x^n$$
   is an integer. *Hint*: use strong induction, and try to express $x^{k+1} + 1/x^{k+1}$ in terms of $x^k + 1/x^k$, $x^{k-1} + 1/x^{k-1}$, and $x + 1/x$.

4. (Nicholas) Proofs by induction are also incredibly useful in proving the correctness of an algorithm. Here is a program that can be used to calculate the square of a number:

Algorithm 1 Square Calculation Function

```
function SQUARE(n)
    S ← 0
    i ← 0
    while i < n do
        S ← S + n
        i ← i + 1
    end while
    return S
end function
```

Prove by induction that this algorithm is correct by showing that after going through the loop k times, S $= k * n$ and i $= k$.

5. (Randy) Suppose that $*$ is a binary operation on A, a function with domain $A \times A$ and codomain A. Prove that if $*$ is associative on three elements i.e. $\forall a, b, c \in A$ we have $a * b * c = (a * b) * c = a * (b * c)$ then $*$ is associative for elements $a_1, ..., a_n \in A$.
Notice that we can write $a * b * c * d$ as $(a * b) * (c * d)$ or $a * ((b * c) * d)$ and many other ways.

6. (John) Prove that for all integers $n \geq 4$, the statement $n! > 2^n$ holds true.

Solutions

1. Base case: the flower of size 1 has $2(1)(1+1) = 4$ stars
   Induction step: Given that the flower of size k has $2k(k+1)$ stars we need to show that the flower of size k+1 has $2(k+1)(k+2)$ stars.
   To turn a flower of size k into a flower of size k into a flower of size $k + 1$ we will replace some zeros with stars. On each row except the first and the last row we will replace two zeros with stars: one star will be to the left of the current leftmost star and one will be to the right of the rightmost star. On the first and the last row we will instead just place on star in the middle.
   As an example here is how we turn a flower of size 2 into a flower of size 3:
   We start with flower of size 2:
   0 0 0 0 0 0 0
   0 0 0 * 0 0 0
   0 0 * * * 0 0
   0 * * 0 * * 0
   0 0 * * * 0 0
   0 0 0 * 0 0 0
   0 0 0 0 0 0 0
   Then we add stars on each row. Newly added stars are marked with "@":
   0 0 0 @ 0 0 0
   0 0 @ * @ 0 0
   0 @ * * * @ 0
   @ * * 0 * * @
   0 @ * * * @ 0
   0 0 @ * @ 0 0
   0 0 0 @ 0 0 0
   Starting with flower of size k and $2k(k+1)$ stars we will add 2 stars to each row except for the first row and the last. Since the flower of size k+1 will be on a $2(k+1) + 1$ by $2(k+1) + 1$ grid we will have $2k+3$ rows. Out of those we will add 2 stars to all but two rows(the first and the last row will only have one star added), resulting in $(2k+1)*2$ additional stars the other two rows will have one star each resulting into more stars. Overall $2k(k+1) + 2(2k+1) + 2 = 2k(k+1) + 4(k+1) = (2k+4)(k+1) = 2(k+1)(k+2)$ So we have shown that given that the flower of size k has $2k(k+1)$ stars then the flower of size k+1 will have $2(k+1)(k+2)$ stars completing the proof.

2. Base Case: n = 1. With 1 pumpkin, there are only $2^1 = 2$ ways to carve its face.
   Inductive Hypothesis: Assume the statement is true for some k, such that there are 2^k ways to carve faces on k pumpkins.
   Inductive Step: k+1 pumpkins have 2^kx2 ways to carve faces. Simplified, this equals to 2^{k+1}, thus proving our inductive hypothesis.

3. Base cases: When $n = 0$, we have $x^0 + 1/x^0 = 2$ and that's an integer. When $n = 1$, it is given that $x + 1/x$ is an integer.

   So assume that the property holds up to some $k \geq 1$, and let's consider $k + 1$. Observe that:

   $$x^{k+1} + 1/x^{k+1} = (x^k + 1/x^k)(x + 1/x) - (x^{k-1} + 1/x^{k-1})$$

   Therefore, by the inductive hypothesis, $x^{k+1} + 1/x^{k+1}$ must be an integer, since every term on the right-hand side is an integer. The inductive step works for every k such that $k - 1 \geq 0$ so that $k - 1$ is non-negative. So we require $k \geq 1$. Therefore, the choice of $n_0 = 1$ is enough for the base cases.

4. Base case: When $k = 0$, we have entered the loop 0 times, and so $S = 0$ and $i = 0$. Thus $S = k * n$ and $i = k$ holds.

   Inductive Hypothesis: Assume the statement is true for some $k = m$. Then $S = m * n$ and $i = m$ after going through the loop m times.

   Inductive Step: Prior to entering the loop for the $m+1$ time, $S = m*n$ and $i = m$. After entering the loop for the $m + 1$ time, $S = m * n + n$ and $i = m + 1$. Hence $S = n * (m + 1)$ and $i = m + 1$. Thus at the end of our $m + 1$ loop, we get our desired results.

   When the algorithm stops at $i = n$, the loop would have been executed n times, and so, $S = n * n = n^2$. Therefore, our algorithm is correct.

5. Base case: We are given that $*$ is associative for three elements.

   Inductive Hypothesis: Assume that for any n elements $*$ is associative.

   Inductive Step: For any expression involving $n + 1$ elements there is a $*$ that is applied last, this $*$ partitions the $n+1$ terms into two expressions each involving n or less elements. By the inductive hypothesis both of those expressions are unique and we replace them by their leftward expansion i.e. $a * b * c * d = a * (b * (c * d))$. Thus every parenthization of $n + 1$ terms is of the form $(a_1 * ... * a_i) * (a_{i+1} * ... * a_{n+1})$

which is equal to $a_1 * (a_2 * ... * (a_n * a_{n+1})...)$. Therefore $*$ is associative on n elements.

6. Base case: For $n = 4$, $4! = 24$ which is greater than $2^4 = 16$.

   Inductive Hypothesis: Assume the statement is true for some integer k≥ 4.
   Inductive Step: We will prove that $(k+1)! \geq 2^{k+1}$ holds true. Observe that: $(k+1)! = (k+1) * k! = (k+1) * 2^k > 2 * 2^k = 2^{k+1}$. Since we know that k$\geq 4$, we know that $k + 1 > 2$. Therefore, $(k+1)! \geq 2^{k+1}$ is true for all integers $n \geq 4$.

Week 9

Recurrences, characteristic equation method, number theory (divisibility and Euclidean algorithm).

1. (Tasmina) For Thanksgiving dinner, there are 84 ounces of turkey and 60 ounces of gravy. The cook wants to create serving trays such that all trays have the same portion of turkey, and all trays have the same portion of gravy, and no food is left over. What is the largest amount of trays the cook can make?

2. (Randy) Show that for any set of 2025 integers, there are two integers such that their difference is a multiple of 2024.

3. Find a closed form for the following recurrence relation, given that $a_0 = 1; a_1 = 1$:
   $a_n = 7a_{n-1} - 12a_{n-2}$

4. (John) Prove that for all $n \geq 1$, $2^{2^n} - 1$ is divisible by 3.

Solutions

1. To solve this problem, you need to find the greatest common divisor (GCD) of 84 and 60, via the Euclidean algorithm. First, divide 84 by 60 to get 1 with a remainder of 24. Then, divide 60 by this remainder of 24 to get 2 with a remainder of 12. Finally, divide 24 by the new remainder of 12 to get 2 with a remainder of 0. Since 0 is the remainder, 12 is the GCD. Thus, the largest amount of plates you can have are 12 plates, with 7 ounces of turkey and 5 ounces of gravy on each.

2. By pigeonhole principle, 2 of them will have the same remainder when divided by 2024 in other words $p \equiv q \pmod{2024}$ so $2024 | p - q$.

3. Using the characteristic equation method, the characteristic equation for this recurrence will be:
   $x^2 = 7x - 12$
   Solving for x we get:
   $x = 3, 4$
   so our recurrence will have a closed form that looks like this:
   $a_n = c_1(3^n) + c_2(4^n)$ We can use the values for a_0 and a_1 to solve for the coefficients:
   $a_0 = 1 = c_0 + c_1$
   $a_1 = 1 = 3c_1 + 4c_2$
   $c_1 = 3; c_2 = -2$
   So our final answer is:
   $a_n = 3(3^n) - 2(4^n) = 3^{n+1} - 2(2^{2n}) = 3^{n+1} - 2^{2n+1}$

4. We can rewrite the statement '$2^{2^n} - 1$ is divisible by 3' as $2^{2^n} - 1 \equiv 0 \pmod 3$.
   Base Case (n=1): $2^{2^1} - 1 = 4 - 1 = 3|3$
   Inductive Step: Assume the statement holds for $n = k$, meaning $2^{2^k} - 1 \equiv 0 \pmod 3$. We will prove that $2^{2^{k+1}} - 1 \equiv 0 \pmod 3$. Note that:
   $2^{2^{k+1}} = 2^{2*2^k} = (2^{2^k})^2$
   $2^{2^k} - 1 \equiv 0(\text{mod}3) \Rightarrow 2^{2^k} \equiv 1(\text{mod}3)$
   $(2^{2^k})^2 \equiv (1)^2(\text{mod}3)$
   $(2^{2^{k+1}}) \equiv 1(\text{mod}3)$
   $(2^{2^{k+1}}) - 1 \equiv 0(\text{mod}3)$
   Thus $2^{2^n} - 1$ is divisible by 3 for all $n \geq 1$.