CSCI 150 Spring 2025 TA's questions

Tasmina, Vladislav, John,
Nicholas, Zach, Zhen Tao,
Nathaniel, Femi, Soren
with help and edits by Saad
Computer Science, Hunter College

Collection 1

Topics: The sum $T_n = 1 + 2 + \ldots + n = n(n+1)/2$ (triangular numbers), planar graphs, Euler's formula for planar graphs $v - e + f = 2$, T_{n-1} is the number of pairs on n objects, generalization of sum to $a + (a+s) + (a+2s) + \ldots + b = \frac{a+b}{2}(\frac{b-a}{s} + 1)$, counting pairs, permutations and $n!$, sum and product notations $\sum$ and $\prod$, manipulation of sum and product notation, splitting sums and nested sums (also seen as nested loops), the addition rule, the product rule.

1. (Saad) Consider the following modified version of T_n:

$$T_n^- = 1 - 2 + 3 - \ldots + (-1)^{n-1}n$$

   where the sign of successive terms alternates. Explore T_n^- for several values of n and conjecture what it's equal to.

2. (Nathaniel) For what values of n is $T_n = n!$ true? How can we be sure there aren't any others?

3. (Saad) Consider the sum $T_{n,m} = n + (n+1) + \ldots + m$, where $m > n$. This is the sum of all integers between n and m inclusive, and it is called a trapezoidal number.
   - Can you imagine and explain why $T_{n,m}$ is called as such?
   - Find a formula for $T_{n,m}$ is terms of n and m.
   - Express $T_{n,m}$ is terms of triangular numbers.

4. (Vlad) Suppose you have an nxn chess board and you want to place a king and another piece such that the king cannot immediately attack the other piece. Use addition rule along with the following procedure to find how many ways there are to accomplish this. We will first choose a tile to place the king and then choose a 2nd tile out of the remaining available tiles.

   (a) Suppose we first choose a corner tile. How many ways are there to choose a corner tile? Given that we have chosen a corner tile, how many ways are there to choose a non adjacent tile? Use the product rule to find the number of ways to choose 2 tiles if the first chosen tile is a corner.

   (b) Suppose instead that we choose an edge tile. How many ways are there to choose an edge tile? Given that we have chosen an edge tile, how many ways are there to choose a non adjacent tile? Use the product rule to find the number of ways to choose 2 tiles if the first chosen tile is an edge.

   (c) Suppose we choose a tile that is not an edge or a corner. How many ways are there to choose a such a tile? Given that we have chosen a tile that is not an edge or a corner, how many ways are there to choose a non adjacent tile? Use the product rule to find the number of ways to choose 2 tiles if the first tile chosen is not a corner or an edge.

   (d) Now you can use the addition rule to find the total number of ways to place a king and another chess piece on the board without the king immediately being able to attack the other piece.

   (e) EXTRA try this with another chess piece replacing the king.

5. (Saad) Come up with a graph with 6 vertices and 9 edges that is planar. Show two ways of drawing it, one with crossings and one without.

6. (Zach) Hunter College is planning on seating people in the auditorium for a performance by some of the students, and there is a group in particular they are trying to seat together of 34 friends and family of some of the performers. If there are 9 seats in the front row, and there are 5 additional seats in each consecutive row heading to the back, what is the row closest to the stage we can seat this group? Come up with a formula (call it $T(i)$) that tells us how many seats are in each row of the auditorium. Use this formula to calculate how many seats are in the row closest to the stage that can accommodate the group.

7. (Zhen) In a technical interview for your dream company Chanel.ai, you are asked to code a function with the following description:

   - the function takes an array, called nums, with size equal to $n+1$
   - the array contains all integers from 1 to n once, but ONE integer has a duplicate
   - your task is to find that duplicate number and return it

   Your solution must be $O(1)$ space complexity (AKA no frequency map, no additional vectors, no addition sets, etc), and $O(n)$ runtime complexity (AKA no nested for-loop). If you don't know how to write a function, perform the mathematical procedure that'll get you the answer.

   ```
   def find_duplicate(nums : list) -> int:

   (or)

   int find_duplicate(int* arr, int size);
   ```

8. (John) Consider the following summation:

   $$S(n) = \sum_{i=0}^{n} (i+1)(a+ib)$$

   Derive an expression for $S(n)$ in terms of n, a, and b.
   Hint: $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

9. (Nicholas) There are 5 books on a shelf: 2 computer science books, 2 math books, and 1 english book. How many ways are there to arrange the 5 books if you want to keep the computer science books together and the math books together?

10. (Tasmina) You are in a class of 15 cs students and 15 math students (including yourself). The professor has announced that everyone must pair up for the final project, but each team must consist of one cs student and one math student. The professor claims that there are 15! ways of making this happen. Explain why. In particular, why is it not 15!15!, which results from permuting each category of students to make up the teams?

11. (Soren)

   (a) How many elements are in the following sequence?

   $$9, 16, 23, \ldots, 100$$

   (b) Using the information obtained in part (a), write this sequence in summation notation. Evaluate its sum.

   (c) Now that we've practiced the fundamentals, we're going to add a twist to this sequence:

   $$2, 7, 9, 14, 16, 21, 23, \ldots, 100$$

   Oh no, they're not uniform intervals! And yet, we can still work with this! How can we express this sequence in terms of summation notation and solve for it?

   (Hints are provided in light-colored text; highlight and zoom in to read)

   Hint: The sequence seems consistent when reading either set of alternating numbers, a property of summations will be useful here.

   (d) (Optional challenge): Can you express this sequence using only one summation?

   (This one is harder than I expected, yet still has a consistent solution. I've provided hints for the method I used to solve, others surely exist)

   Hint 1: We will need three terms with three coefficients: a constant term, a term multiplying i, and a term with an alternating sign. The idea behind this third term is that it alternates between adding to and subtracting from the term; do you know how to do this?

   Hint 2: The alternating sign is done by multiplying the third coefficient by $(-1)^{i+1}$, or some other term involving i

   Hint 3: Solve for the three coefficients using a system of equations for the first three numbers of the sequence.

Solutions

1.

$$T_1^- = 1$$
$$T_2^- = 1 - 2 = -1$$
$$T_3^- = 1 - 2 + 3 = 2$$
$$T_4^- = 1 - 2 + 3 - 4 = -2$$
$$T_5^- = 1 - 2 + 3 - 4 + 5 = 3$$
$$T_6^- = 1 - 2 + 3 - 4 + 5 - 6 = -3$$

It looks like $T_n^- = (-1)^{n-1}\lceil n/2 \rceil$.

2.

$$T_1 = 1 = 1!$$
$$T_3 = 3 + 2 + 1 = 6 = 3! = 3 * 2$$

We can be confident that there are no other values where this is true because $n!$ grows far faster than T_n; so T_n will never "catch up". Later we will learn to formally prove statements like this.

3. Start with n dots on the first row, followed by $n+1$ dots on the second, and so on until m dots. This creates a trapezoidal shape. As we have seen in class, $T_{n,m} = \frac{n+m}{2}(m - n + 1)$. We can also express it as the difference of two triangular numbers: $T_{n,m} = T_m - T_{n-1}$, since

$$T_{n,m} = n+(n+1)+\ldots+m = (1+2+\ldots+m)-[1+2+\ldots+(n-1)] = T_m - T_{n-1}$$

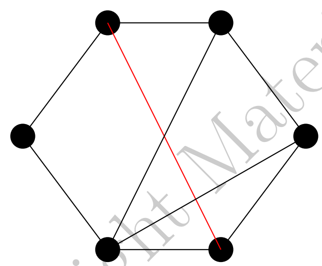
4. (a) There are 4 corners so 4 ways to choose a corner tile. If a king is placed in a corner he can attack 3 other tiles you also can't place the other piece where the king is placed, so that leaves $n^2 - 4$ tiles. By product rule there are $4(n^2 - 4)$ ways to place the pieces if the king was placed in a corner.

   (b) There are $4(n - 2)$ so $4(n - 2)$ ways to choose an edge tile. If a king is placed in an edge tile he can attack 5 other tiles you also can't place the other piece where the king is placed, so that leaves $n^2 - 6$ tiles. By product rule there are $4(n - 2)(n^2 - 6)$ ways to place the pieces if the king was placed in an edge tile.

(c) There are $(n-2)^2$ tiles left you can find that either by inspection or by subtracting results from the previous two parts from n^2. In this situation a king can attack 8 tiles and you can't place another piece on the tile that the king is at leaving $n^2 - 9$ tiles available. By product rule there are $(n-2)^2(n^2-9)$ ways in this case.

(d) By addition rule, since the cases we computed in the previous parts are distinct, adding up the answers from the previous parts gives us the final answer.

5. Here's a graph with 6 vertices and 9 edges that is planar. Move the red edge from the "inside" to the "outside" to eliminate crossings.



6. A formula for the number of seats in each row of the auditorium is $T(i) = 9 + 5(i-1)$, where 5 is the number of steps and i is the row number. According to this forumla, $i = 6$ is the row closest to the stage that can accommodate our group, because $T(6) = 9 + 5(5) = 25 + 9 = 34$. So, there are 34 seats in the row.

7. We know the sum of numbers from 1 to n is given by $n(n+1)/2$. In the array, there's one duplicate, let's call it x. The sum of all the integers in that array will then be $n(n+1)/2 + x$. We will subtract $n(n+1)/2$ from the sum of the array, and we'll get the solution. The first is a python solution, and the 2nd is C++.

```python
def find_duplicate(nums : list) -> int:
    sum_of_array = 0
    n = len(nums)-1 # number of unique integers

    for num in nums:
        sum_of_array += num

    return sum_of_array - (n*(n+1)/2)
```

```c
int find_duplicate(int* arr, int size) {
    int sum_of_array = 0;
    n = size - 1 // number of unique integers
    for (int i = 0; i < size; i++) {
        sum_of_array += arr[i];
    }
    return sum_of_array - (n*(n+1)/2);
}
```

8. The first step in deriving an expression for $S(n) = \sum_{i=0}^{n}(i+1)(a+ib)$ is to split the sums. You can either expand the terms and then combine the separate sums, or you can realize that $S(n) = \sum_{i=0}^{n}(i+1)$ is already in an ideal form for splitting sums, and simply distribute the second factor to get

$$S(n) = \sum_{i=0}^{n} a(i+1) + \sum_{i=0}^{n}(ib)(i+1) = a\sum_{i=0}^{n}(i+1) + b\sum_{i=0}^{n}(i^2+i)$$

We then know that

$$a\sum_{i=0}^{n}(i+1) = a\frac{(n+1)(n+2)}{2}$$

and

$$b\sum_{i=0}^{n}(i^2+i) = b\Big[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\Big] = b\frac{2n(n+1)(n+2)}{6}$$

Adding the two expressions will give us

$$S(n) = a\frac{(n+1)(n+2)}{2} + b\frac{n(n+1)(n+2)}{3}$$

9. Pretend that we didn't care about keeping the genres of the books together. Then, the answer would be 5!. Where did the 5 come from? It's from the number of objects that we have to permute. In this harder version of this question, since we want to keep the genres of the books together, instead of permuting on the number books, we will permute on the number of genres. This way we guarantee that books of the same genre will stay together. The number of permutations of the genres is 3!. Then, within the genres themselves, there are ways that we can permute the books. There are 2! ways to permute the 2 computer science books, 2! ways to permute the 2 math books, and 1! way to permute the english book. Therefore, our final result is 3! * 2! * 2! * 1! = 6 * 2 * 2 * 1 = 24.

10. Place all cs students in some order. Now permute all the math students in 15! ways, and assign the ith cs student to the ith math student. This will produce all possible ways of teaming up. If instead we permute all cs students, and permute all math students, then assign them accordingly, we overcount. For example, let's consider a smaller instance of the problem where we have three cs students $(1, 2, 3)$ and three math students (a, b, c). Observe that multiple permutations of both can lead to the same teams:

$$(1, 2, 3), (a, b, c)$$
$$(1, 3, 2), (a, c, b)$$
$$(2, 1, 3), (b, a, c)$$
$$(2, 3, 1), (b, c, a)$$
$$(3, 1, 2), (c, a, b)$$
$$(3, 2, 1), (c, b, a)$$

So we overcount here by 3!.

11. (a) Observing this is a linear sequence, we can count the terms by adjusting the sequence such that each term is a multiple of the step value:

$$1s, 2s, 3s, 4s, \ldots, \text{n}s$$

Noticing the step value of our sequence is 7, we can subtract each term by 2 to obtain the following (note the number of elements did not change):

$$7, 14, 21, \ldots, 98$$

$98/7 = 14$, and we have our answer. $n = 14$

(b) Contrary to what the phrasing of the question implies, we do not need summation notation to solve for the sum of this sequence (Chapter 0 Section 3 has an explanation). We will continue with finding summation notation and then solve for the sum using that.

Now knowing the length of our sequence, we need to find the summand (right side of the summation notation) and properly index it.

We see that the intervals are by 7, so this must be the coefficient of the index variable. (i.e. $7i + \dots$)

We can also see the the entire sequence is offset by 2, so this must be our constant.

Thus, we have our summation (alternatives are also listed):

$$\sum_{i=1}^{14} \Big(7i + 2\Big) \tag{1a}$$

$$\sum_{i=0}^{13} \Big(7(i+1) + 2\Big) \tag{1b}$$

$$\sum_{i=0}^{13} \Big(7i + 9\Big) \tag{1c}$$

We will work with expression 1a.

Using properties of summations: we can separate the summation of a sum to be the sum of the summations, and factor out the coefficient:

$$7\sum_{i=1}^{14} i \quad + \quad \sum_{i=1}^{14} 2$$

We can solve this knowing how to solve for the sum of the sequence of the first n natural numbers.

$$7{\cdot}105 + 28$$
$$735 + 28$$
$$763$$

(c) The trick here is to separate this sequence into two sequences, which are uniform. This will utilize what we've practiced in parts a and b.

$$\sum_{i=0}^{14}(7i+2) \quad + \quad \sum_{i=1}^{14} 7i$$
$$\sum_{i=0}^{14} 7i \quad + \quad \sum_{i=0}^{14} 2 \quad + \quad \sum_{i=1}^{14} 7i$$
$$735 \quad + \quad 30 \quad + \quad 735$$
$$1500$$

(d) We are looking for a summation of the form:

$$\sum_{i=0}^{28}\Big(x + yi + z(-1)^{i+1}\Big)$$

We can solve this using a system of equations, modelled after the first three terms of the sequence:

$$\begin{aligned} x \quad\;\; - z &= 2 \\ x + y + z &= 7 \\ x + 2y - z &= 9 \end{aligned}$$

Solving for the three variables, we get:

$$x = \frac{11}{4} \qquad y = \frac{14}{4} \qquad z = \frac{3}{4}$$

Subbing these into the summation, we get our final answer:

$$\sum_{i=0}^{28}\Big(\frac{11}{4} + \frac{14}{4}i + \frac{3}{4}(-1)^{i+1}\Big)$$

We can verify this creates the desired sequence through substitution.

There is also a trigonometric solution to this, found by Zhen Tao:

$$\sum_{i=1}^{29}\left(\frac{7}{2}i - \frac{3}{2}\left|\sin\frac{\pi i}{2}\right|\right)$$

The trigonometric function, combined with the absolute value function, serves to create a factor alternating between 0 and 1.

Collection 2

Topics: Addition rule, multiplication rule, 4 kinds of selection: unordered without repetition, ordered without repetition, unordered with repetition, ordered with repetition, k-permutations, k-combinations, problems with binary strings, problems with words, handshake lemma, counting in general...

1. (Soren) We can look to Figure 1 for this problem.

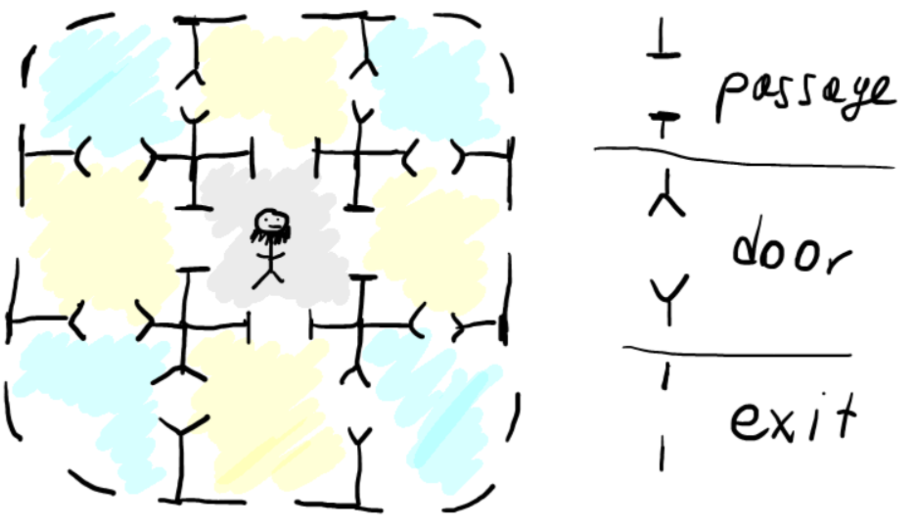


Figure 1: A dwarf that needs to find his way out

   (a) A dwarf named Urist was enjoying his afternoon drink in the tavern when, all of a sudden, he was struck by a fey mood and needs to find his way outside the fortress for materials.
   Assuming Urist cannot pass through any passage or door more than once, nor can he pass through two passages or two doors, how many paths can Urist take to exit the fortress?

   (b) (Optional challenge) We shall remove a restriction: Urist can use as many doors as he likes, but cannot enter the same room twice (including the central room, which he started having entered). Succinctly rephrased: "How many different paths can Urist take to leave the fortress if he cannot enter any room more than once and cannot re-enter the center room?"

2. (Nathaniel) **The five great names of Pharaoh**:

   A new pharaoh has risen over the Kingdom of Upper and Lower Egypt and must choose a multi-part regnal name. In addition to the personal name he already has, he must choose four new names:

   A Horus Name, A Nebty Name, A Throne Name, And a Golden Horus Name.

   Each name **must be 3 hieroglyphs long**. There are **763 hieroglyphs** for Pharaoh to choose from, but he must abide by the following restrictions:

   (a) The Horus Name can be any combination of 3 hieroglyphs;

   (b) The Nebty Name must contain only hieroglyphs not used in the Horus Name and cannot repeat them;

   (c) The Throne Name must rhyme with his personal name (i.e. end with the same hieroglyph);

   (d) The Golden Horus Name must contain only the same hieroglyphs as the Horus Name, in any order.

   How many ways can Pharaoh choose his name?

3. Given a full shuffled deck of cards(4 suits, 13 cards of each suit) what is the probability of:

   (a) Drawing 5 cards from the deck and getting a royal flush of hearts(A, K, Q, J, 10 of hearts in any order).

   (b) Drawing 5 cards and getting any royal flush(A, K, Q, J, 10 of the same suit in any order)

   (c) Drawing 5 cards and getting a royal flush then drawing 5 more cards and drawing another royal flush.

   (d) Drawing 10 cards and getting 2 royal flushes(The answer should be different from the previous part)

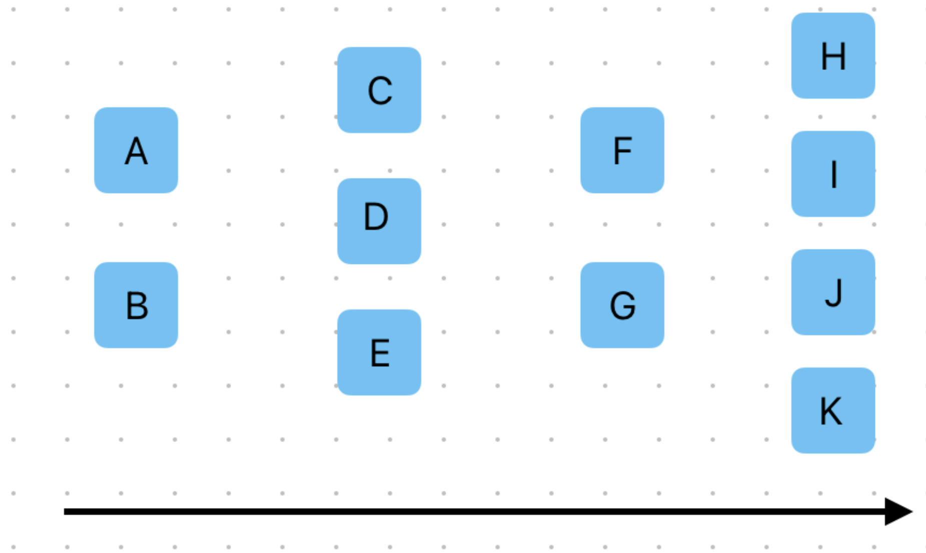
   (e) Discarding the top 3 cards of the deck, then drawing 2 cards, then discarding 8 cards, then drawing 3 cards and have the cards that were drawn make up a royal flush?

   (f) Would the answer to any of the previous change if instead of drawing cards from the top of the deck you alternated drawing from the top and the bottom of the deck.

For each part of this question, assume that the cards drawn from the previous part are reshuffled into the deck.
Hint: To find the probability, find how many ways there are to shuffle the deck given that the top cards in the deck would satisfy the condition outlined in the part. Then divide that number of ways over the total number of ways to shuffle a deck.

4. (Tasmina) You have been transported to a fantasy world where you are tasked with creating a city. This city contains land masses and bridges that connect two land masses to another. You would like to design this city such that there is an odd number of land masses, with an odd number of bridges attached to each of them. Explain if this is possible and identify what principle you used.

5. (Zach)



You are tasked with counting how many paths there are starting from the left side of this diagram to the right side by starting at either A or B and then traveling through C, D, or E, and then F, or G, and then H, I, J, or K. The only other rules are that if you're coming from A, you can only travel to C or D; if you're coming from B, you can only travel to E; if you're coming from F, you can only travel to H or I; if you're coming from G, you can only travel to J or K.

Following these rules, count how many paths there are using the addition and multiplication rules.

6. (Nicholas) You are located on the bottom left corner of an 8 x 8 grid. The end destination is the top right corner of the same grid. Given that you can only move left or up:

   (a) Count the number of paths that you can take to get to the end destination.

   (b) Count the number of paths that you can take to get to the end destination such that you change directions an odd number of times.

   Bonus: Is it more likely to get to the end destination with an odd number of direction changes or an even number of direction changes?

7. (Femi) Given there is a group of 365 people, each born on a different day of the year 2003:

   (a) What is the likelihood that 2 randomly selected people were born within a day of each other?

   (b) How about the likelihood that they were born within a week of each other?

Solutions

1. (a) We can observe that there are two ways to leave the fortress, via exits in:

   A yellow room (passage → exit) These are simply countable: there are only 4 ways to exit using this method

   A blue room (passage → door → exit) These *could* be counted manually, but this would be tedious. Noticing the uniformity of the fortress, we can apply the multiplication rule to find the number of paths. For every path there are three decisions:

   i. Choose a passage (there are 4 possible choices)
   ii. Choose a door (there are 2 possible choices)
   iii. Choose an exit (there are 3 possible choices)

   More specifically defining the application of the multiplication rule earlier: seeing as these choices are independent of each other (what Urist chooses in (i) does not affect the number of possible choices for (ii) or (iii); and vice versa), we can multiply $4 \cdot 2 \cdot 3$ to obtain 24 possible paths.

   Now the important question for every usage of the multiplication rule: did we overcount? We are not able to permute our choices because the choices involve choosing distinct objects; therefore, it is impossible for us to reach the same path by more than one decision tree. We did not overcount. We do not need to divide 24 by anything, since every path is unique.

   Seeing as the two categories above are mutually exclusive (Urist can only exit in a yellow room or a blue room; not both), we can apply the addition rule to add the possible paths. $24 + 4 = 28$. We have our answer: Urist can exit the fortress via 28 different paths.

   (b) (Optional challenge) When we want to apply the multiplication rule to find a countable quantity, one method is to look for patterns that repeat. We do this by recognizing a smaller countable quantity, then multiplying that quantity by the factor of repetitions (ensuring that they are in fact exact duplicates). This solution uses this method.

   Observe that, after choosing a passage and an initial door (each combined choice of which is mutually exclusive from any other choice of passage and initial door), there are is a finite chain of seven possible rooms to exit out of. For the sake of example, let's choose a passage+door choice of UL (up, then left). Therefore, one can draw a path through the seven rooms and simply count every possible exit that can result from the initial choice of UL:

$$3 + 1 + 3 + 1 + 3 + 1 + 3$$
$$15$$

There are 15 possible exits resulting from UL. Observing that there are 8 possible choices of passage+door ($4 \cdot 2$; UL, UR, RU, RD, DL, DR, LU, LD), we multiply $15 \cdot 8$ to get 120 possible exits. Add the 4 exits that can be taken without passing through a door, and we get our total of 124 possible exit paths.

Another method is looking backwards by asking the two questions:

- How many paths exist for any blue-room exit to the center?
  - For any blue-room exit, observe that there is exactly one path for every passage+door choice above. This yields 8 paths for every blue-room exit.
- How many paths exist for any yellow-room exit to the center?
  - For any yellow-room exit, observe that there is only 1 path for that side's passage choice (which would be exiting before choosing a door) and 6 paths for the other choices of passage. This yields 7 paths for every yellow-room exit.

Having obtained the answers to these questions, we can arrive at the solution by multiplying the possible paths for every exit by the number of its color-corresponding exits, then adding the paths for both yellow and blue. Counting 4 yellow exits and 12 blue exits:

$$4 \cdot 7 + 12 \cdot 8$$
$$124$$

We have found our same solution of 124 possible exit paths. Finding the same quantity through two distinct methods is referred to as a combinatorial proof by double counting, and it strengthens our conviction our solution is indeed correct. An excellent technique for verifying our solutions on exams!

2. (a) Each hieroglyph in the Horus Name can be chosen 763 ways. Apply the product rule and we have 763^3 ways to choose the name.

   We must now break the set of possible names down into three disjoint sets: the sets of names where the Horus Name has one unique hieroglyph, two unique hieroglyphs, and three unique hieroglyphs. For brevity, we will denote these cases as $H = 1$, $H = 2$, and $H = 3$.

   If $H = 1$, there are 763 ways to make the Horus Name.

   If $H = 2$, there are $\binom{763}{2} \cdot 6$ ways to make the Horus Name: using the product rule, we multiply the number of ways to choose the two hieroglyphs by the number of ways to arrange the ones we've chosen (for example *baa*, *aba*, *aab*, *abb*, *bab*, and *bba*).

   If $H = 3$, there are $\frac{763!}{(763-3)!}$ ways to make the Horus Name. This is the same as choosing the name without repeating hieroglyphs.

   You can verify that this covers the set of all possible Horus Names by observing that

$$763 + \binom{763}{2} \cdot 6 + \frac{763!}{(763-3)!} = 763^3$$

   (b) If $H = 1$, we can form $\frac{762!}{(762-3)!}$ Nebty Names. If $H = 2$, this drops to $\frac{761!}{(761-3)!}$. If $H = 3$, it drops to $\frac{760!}{(760-3)!}$.

   (c) Since the personal name was already chosen when Pharaoh was born, we only need to choose two hieroglyphs for 763^2 combinations.

   (d) If $H = 1$, we can only form 1 Golden Horus Name. If $H = 2$, we can form 3 Golden Horus Names (for example *abb*, *bab*, and *bba* once the letters are known). If $H = 3$ we can form $3! = 6$ Golden Horus Names.

Now we must apply the product rule to get the total size of each set.

The size of the set of possible names where $H = 1$ is $763 \cdot \frac{762!}{(762-3)!} \cdot 763^2 \cdot 1$

The size of the set of possible names where $H = 2$ is $\binom{763}{2} \cdot 6 \cdot \frac{761!}{(761-3)!} \cdot 763^2 \cdot 3$

The size of the set of possible names where $H = 3$ is $\frac{763!}{(763-3)!} \cdot \frac{760!}{(760-3)!} \cdot 763^2 \cdot 6$

Now we use the addition rule to add up the disjoint set. So there are

$$\frac{762!}{(762-3)!} \cdot 763^3 + \binom{763}{2} \cdot 18 \cdot \frac{761!}{(761-3)!} \cdot 763^2 + \frac{763!}{(763-6)!} \cdot 763^2 \cdot 6$$

ways for Pharaoh to choose a name.

3. In order to get probability for any of the subquestions we should first calculate the number of ways to shuffle a deck. There is 52 distinct cards in a card deck so there is 52! ways to shuffle it.
   Here is a way to represent a deck: $[K\clubsuit, 9\heartsuit, 10\diamondsuit...]$ in this representation cards to the left of the list are closer to the top of the deck, so drawing from the deck represented by this list you would first draw a King of clubs, then a 9 of hearts, then a 10 of diamonds etc...

   (a) Our deck needs to be of the form: $[R\heartsuit, R\heartsuit, R\heartsuit, R\heartsuit, R\heartsuit, X, X, ...X, X]$ where R represents one of the ranks needed for a royal flush(10, J, Q, K, A) and X represents any other card. We can find the number of ways using the following procedure:

      i. Order the 5 Cards that make up the royal flush of hearts: 5!
      ii. Order the remaining 47 cards: 47!

      This results in $5! \cdot 47!$ ways, so the probability of this happening is $\frac{5!47!}{52!}$

   (b) We follow a similar procedure to the previous part but with one extra step:

      i. Pick a suit: $\binom{4}{1} = 4$
      ii. Order the 5 Cards that make up the royal flush of the chosen suit: 5!
      iii. Order the remaining 47 cards: 47!

      This results in $4 \cdot 5! \cdot 47!$ ways, so the probability of this happening is $\frac{4 \cdot 5!47!}{52!}$

   (c) Now we also need to account for picking two suits and ordering them. Due to the way the question is phrased the first 5 cards must be of the same suit, followed by 5 more cards of a different suit:

      i. Pick two suits: $\binom{4}{2} = 6$

ii. Order the two chosen suits: $2! = 2$
iii. Order the 5 Cards that make up the first royal flush : 5!
iv. Order the 5 Cards that make up the second royal flush : 5!
v. Order the remaining 42 cards: 42!

This results in $12 \cdot 5!5!42!$ ways, so the probability of this happening is $\frac{12 \cdot 5!5!42!}{52!}$

(d) Unlike part c we are now allowed to draw the 10 necessary cards in any order. For example you can have a deck like $[K\heartsuit, A\heartsuit, 10\clubsuit, A\clubsuit, Q\heartsuit, 10\heartsuit, J\heartsuit, K\clubsuit, Q\clubsuit, J\clubsuit, ...]$ and it would be a valid shuffle even though you don't draw a royal flush every 5 cards:

i. Pick two suits: $\binom{4}{2} = 6$
ii. Order the 10 cards that make up the royal flushes: 10!
iii. Order the remaining 42 cards: 42!

This results in $6 \cdot 10!42!$ ways, so the probability of this happening is $\frac{6 \cdot 10!42!}{52!}$

(e) This part asks us for all possible deck shuffles where the 4th, 5th, 14th, 15th and 16th cards make up a royal flush. since we can still rearrange the 5 cards among themselves and the rest of the cards can be arranged in any way the answer should be the same as part b.

(f) Similarly to part e it doesn't matter what way we draw the cards the procedure for constructing valid shuffles will remain the same. Thereofore none of the answers to the previous parts would change.

4. It is not possible to have an odd number of land masses with an odd number of bridges to them. Think about the handshake lemma. When there is an odd number of land masses with an odd number of bridges, you will be adding an odd number of degrees an odd number of times. Odd x Odd = Odd. This means that you will end up with an odd total of degrees, which violates the handshake lemma, which states that it must be even (to be twice the number of edges).

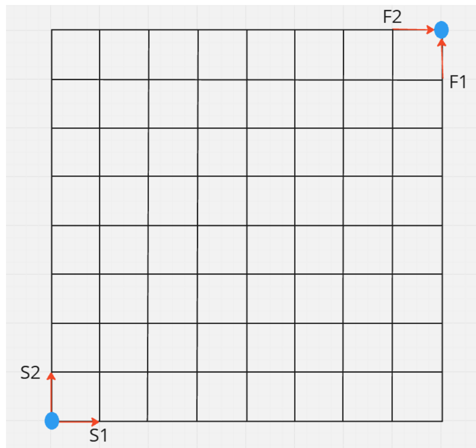
5. To count all the possible paths, we first notice that the set of paths starting at A and the set of paths starting at B are disjoint (because no two paths are exactly the same when the first node is different), so we will use the addition rule to add these two sets at the end. Further, notice that the set of paths starting at C and the set of paths

starting at D are disjoint for the same reason. So we use the addition rule for those as well. Next, to count the paths from F to either H or I we multiply $1 \cdot 2$ - we start with one choice (F) and then have two choices (H or I). The same works when we're traveling from G to J or K. So, putting it all together, for A we have: For paths starting at C: $2 \cdot 1 + 2 \cdot 1$ (because C can go to either F or G), and for paths starting at D (coming from A) we have: $2 \cdot 1 + 2 \cdot 1$ (using the same logic as before). Because the paths starting at C and D are disjoint, we add them getting: $(2 \cdot 1 + 2 \cdot 1) + (2 \cdot 1 + 2 \cdot 1)$, yielding the number of paths starting at A. Now, for B, it is much simpler since instead of C and D we just have E. So, we add the number of paths starting at F and G to get the number of paths starting at B: $2 \cdot 1 + 2 \cdot 1$. Finally we add the disjoint sets of paths starting at A and paths starting at B using the addition rule to end up with: $[(2 \cdot 1+2 \cdot 1)+(2 \cdot 1+2 \cdot 1)]+(2 \cdot 1+2 \cdot 1) = 12$ total paths. 12 paths is the answer.

Another way is this: If we start at A, we can imagine making a path in 3 phases, where each phase can be done in 2 ways: In phase 1, we can go to C/D, so that's 2 ways. In phase 2, we can go to F/G, so that's 2 ways. In phase 3, and regardless where we landed in phase 2, we can also go to H/I or J/K, and that's 2 ways as well. Using the product rule, we have $2 \cdot 2 \cdot 2 = 8$ ways starting at A. The same principle of the product rule can be used to figure out that we have $1 \cdot 2 \cdot 2$ ways starting at B. By the addition rule, we have 8+4=12 ways in total.

6. (a) Treat your starting location as (0,0) and the end destination as (8,8). Since we can only move left and up, to get from (0,0) to (8,8), we will need to perform some combination of 8 Ups and 8 Lefts. The total number of combinations is equal to $16!/(8! \cdot 8!)$.

   (b) Observe that if you decide to make Left your first move, then after the first direction change, you will be going up. After a second, you will be going left, and after a third, you will be going up again. An odd number of direction changes results in you going up, requiring that your final move is one that makes you go up. Through the same observation, if you make Up your first move, an odd number of direction changes results in you going left, requiring that your final move is one that makes you go left. Using this information, the number of ways to get to end destination using an odd number of direction changes is the same as the number of paths from s_1 to f_1 and s_2 to f_2.



To get from s_1 to f_1, we will need to perform some combination of 7 Lefts and 7 Ups. Likewise, To get from s_2 to f_2, we will also need to perform some combination of 7 Lefts and 7 Ups. Therefore, the total number of ways to get to the end destination using an odd number of direction changes is $14!/(7! \cdot 7!) + 14!/(7! \cdot 7!)$.

Bonus: The probability of getting to the destination using an odd number of direction changes is $\frac{\text{number of paths with odd number of direction changes}}{\text{total number of paths}}$. This is the result from part b divided by the result from part a. This turns out to be $\frac{8}{15}$. Since there can only be either an odd number of direction changes or an even number of direction changes, this means that the probability of getting to the end with an even number of direction changes is $1 - \frac{8}{15}$ or $\frac{7}{15}$. Alternatively, you can find this probability by finding the number of even direction changes using the same ideas. Therefore, it's actually more likely to use an odd number of direction changes.

7. (a) There are a total of $\binom{365}{2}$ possible pairs of people that can be selected. Of these, 364 are of two people born within a day of each other. Thus, there is a $364/364/\binom{365}{2} = 364/\frac{365!}{363!2!} = \frac{2}{365}$ chance.

   (b) There is the same total amount of possible pairs: $\binom{365}{2}$. The amount of pairs born within a week of each other can be counted in many ways. There are 6 possible pairs involving January 1 that are within a week of it: January 2-7. If you only count the days after the date being checked to avoid repeating pairs, the same can be assumed for every day up to and including December

25. December 26, however, has 5 pairs because December 31 is the last day of the year. December 27 has 4, 28 has 3, 29 has 2, and 30 has 1. This makes the total amount of pairs: 2169. So, the answer would be $2169/\binom{365}{2}$.

Collection 3

Sets, subsets, power set, sum of binomial coefficients, functions, onto, one to one, bijection. More counting examples, counting using bijections, counting multisets (unordered selection with repetition), integer solutions.

1. (John) In order to prepare for a movie screening, Oswald the Octopus wants to clean his tentacles (8 of them) to present himself professionally. Tentacle i has i suction cups and cleaning each suction cup takes 1 minute. Every time he jumps from a tentacle with a lower number of suction cups to a tentacle with a higher amount he takes an extra 10 minutes to adjust the cleaning device. Oswald wants to finish cleaning within an hour in order to attend the screening on time. Say that the cleaning device is initially configured to clean a tentacle with 0 suction cups, then in how many ways can Oswald choose to clean his tentacles and still make it in time for his appointment?
   Hint: There is an initial 10 min adjustment, and at least $1+2+\ldots+8 = 36$ min of cleaning, for a total of 46. Oswald can afford just one extra adjustment time of 10 min. This means that the order by which he will clean the tentacles must contain no more than one jump from a lower number to a higher number of suction cups.

2. (Nicholas) How many subsets of the set $\{1, 2, ..., 25\}$ have the property that the sum of its elements will be greater than 162?

   Hint: Find a bijection between all such subsets and their complements.

3. (Vlad) Let $f(x)$ be a map (a map is like a function but one input is allowed to have several outputs) that maps the set of real numbers to some subset of the real numbers. Assume that you only have access to a graph of f on a coordinate plane (with inputs x denoted on the horizontal axis and the outputs y denoted on the vertical axis). By only looking at the graph of f how can you prove if the following statements are true?

   (a) f is not a function.
   (b) f is not one-to-one.
   (c) f is not a onto.
   (d) f is not a bijection.

   Hint: To solve part a you could look at "vertical line test" from your other math classes and think about why it works.

4. (Tasmina) Mathematicians have recently discovered a new type of set that can either include an element, hide an element, or exclude an element. A hidden element is denoted by a subscript of H. Given this new type of set, what is the new formula for the power set of any such set with n elements? *Hint*: Start with the set $[1, 2, 3]$ and write all possibilities. Do you see a pattern?

5. (Zach) Say that the Dolciani Math Learning Center at Hunter College has 20 tutors and 15 students, thus having a 3:4 student-to-teacher ratio. If we create a function that assigns tutors to the 15 students, what can we say about the function in the following situations?

   (a) Only 5 tutors are working today. So, 3 students are working with each tutor.

   (b) Every tutor is working today, and each is assigned to a student.

   (c) Today, only 15 tutors are working and each is assigned to a student.

   (d) If we can create a bijection from the set of 15 students to the set of 20 tutors, how can it be done? If we cannot, explain why.

6. (Nathaniel) What is the sum of the coefficients of $(x + y)^4 * (j + i)^3$? Solve using two different methods (and without writing out the entire product).

7. (Soren) We're at a grocery store, and there is a self-serve area for a variety of nuts. There are five types of nuts (peanuts, cashews, walnuts, pecans, hazelnuts) and three varieties of each type of nut (plain, salted, roasted). We want to create our own nut mix by selecting 200 individual kernels of nuts and placing them in a bag (though the mixing happens after purchase, of course). How many different nut mixes can we create?

   (A hint[1] is available in the footnotes, recommended after some thought)

[1] In my opinion, this question is in the style of the test questions: maybe initially obtuse, though clearly familiar after some understanding. Framing the information from the question into one of the discrete math models we've learned is the hard part. See if you can make sense of the quantities and what they represent relative to each other.

Solutions

1. We are looking for all permutations of $a = (1, 2, \ldots, 8)$ such that there exists at most one i that satisfies $a[i-1] < a[i]$, let's call this a jump. So we are looking for all permutations that have at most one jump. Let's generalize this to n instead of 8. First, there is one permutation that has no jumps at all, that's $(n, n-1, n-2, \ldots, 1)$. So let's count the permutations that have exactly one jump. Assume the jump occurs at index i, when $a[i] = j$. So we have $a[i-1] < j$ and $a[i] = j$. There are $n - i$ elements that follow j in the permutation, and they must all be in decreasing order, so they are all less than j. In addition, there must be another element less than j to its left. Since there are exactly $j - 1$ elements less than j, it must be that $n - i < j - 1$ (so $i \geq n - j + 2$). We need to choose $n - i$ elements from $j - 1$ to place on the right of j (all in decreasing order). The rest of the elements are also placed in decreasing order on the left of j, which also guarantees that $a[i-1] < a[i]$. So the number of ways we could do that is:

$$\sum_{j=1}^{n} \sum_{i=n-j+2}^{n} \binom{j-1}{n-i}$$

$$= 0 + \binom{1}{0} + \Big[\binom{2}{0} + \binom{2}{1}\Big] + \Big[\binom{3}{0} + \binom{3}{1} + \binom{3}{2}\Big] + \ldots + \Big[\binom{n-1}{0} + \ldots \binom{n-1}{n-2}\Big]$$

Using the Binomial theorem, this is $(2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \ldots (2^{n-1} - 1) = [2^0 + 2^1 + \ldots + 2^{n-1}] - n = 2^n - 1 - n$. Therefore, we get $2^n - n$ after adding the permutation with no jumps. When $n = 8$, we have $2^8 - 8 = 256 - 8 = 248$.

2. Let S be the set of all subsets that add up to more than 162, and let T be the set of all subsets that add up to at most 162. We will show that there is a bijection $f : S \to T$, as follows:

$$f : S \to T$$

$$f(x) = \{1, 2, \ldots 25\} - x$$

Therefore, if $x \in S$, then $y = f(x)$ is its complement. To verify that $f(x)$ is indeed in T, observe that if x adds up to more than 162 (at least 163), then its complement adds up to at most $(1+2+\ldots+25) - 163 = 325 - 163 = 162$.

f is onto: Let $y \in T$ be a subset that adds up to at most 162. Consider $x = \{1, 2, \ldots, 25\} - y$. Obviously, $f(x) = y$. Now x adds up to at least $1 + 2 + \ldots + 25 - 162 = 325 - 162 = 163$. Therefore, $x \in S$.

f is one-to-one: Observe that if $f(x_1) = f(x_2)$, then the complement of x_1 and the complement of x_2 are the same. This means $x_1 = x_2$.

Therefore, we have shown that we can create a bijection between these two sets. This gives us the crucial information that the number of subsets with a sum greater than 162 is equal to the number of subsets with a sum less than or equal to 162. Since a subset can only either have a sum greater than 162 or a sum less than or equal to 162, these subsets make up all possible subsets. The total number of subsets is 2^{25}. So the number of subsets with a sum greater than 162 is equal to $\frac{2^{25}}{2} = 2^{24}$.

3. (a) To show that f is not a function we just need to show that f takes one input to two or more outputs. On a graph this would mean that for a particular value of x we can draw a vertical line at that x and if the line intersects the graph in more than one spot then x is mapped to two different outputs. EX: a circle on the coordinate plane is not a function but a semicircle cut horizontally is.

   (b) To show that f is not one-to-one we need to show that multiple inputs map to the same output. So for a particular output y if we can draw a horizontal line through y such that it intersects the graph in more than one point that would show that f is not one-to-one. EX: The tangent function is not one-to-one, but $f(x) = x$ is.

   (c) To show that a function is not surjective (onto) we need to find an output y to which no x can be mapped using $f(x)$. So if there is an output y such that drawing a horizontal line through it will result in no intersections with the graph then f is not onto. EX: The $f(x) = \sin(x)$ function is not onto but $f(x) = x * \sin(x)$ is.

   (d) To prove f is not a bijection we can simply prove one of the condition in parts b and c since to be a bijection a function has to be both one-to-one and onto.

4. To understand this problem, review the definition of a power set. It is 2^n because each element has 2 possibilities: inclusion or exclusion in a subset. With the new hidden option, there are now 3 possibilities, so the answer is 3^n instead.

5. (a) The function is not one-to-one, because we have the same tutor assigned to more than one student. (i.e. f(x) = f(y) does not imply x=y)

   (b) The function is not actually a function, since there are 20 tutors and only 15 students and some students must be assigned more than one tutor. (i.e. f(x) takes on more than one value for some x in the domain)

   (c) The function is not onto since there are 5 tutors not assigned to any student. (i.e. some elements of the co-domain don't have an element of the domain that maps to it)

   (d) A bijection is not possible since there are more tutors than there are students. It is impossible to assign a unique tutor to each student. (i.e. the sizes of the domain and co-domain are not equal)

6. (a) First, we can solve this by calculating and summing the coefficients of each term and then multiplying. This gives us:

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

$$16 * 8 = 128$$

   (b) We can also solve this by remembering that the sum of the coefficients is the same as the value of the equation when every variable is equal to 1. Therefore

$$(x + y)^4 * (j + i)^3 = (1 + 1)^4 * (1 + 1)^3 = 2^4 * 2^3 = 2^7 = 128$$

   Pascal's Triangle in the next section will give us a third way to check this answer.

7. So this is an integer partition problem of 200 items into 15 categories (5 types of nuts × 3 varieties). Observing this, the rest is using the

integer partition technique we've been taught:

$$x_1 + x_2 + \ldots + x_{14} + x_{15} = 200$$
$$k = 200$$
$$n = 15$$
$$\underbrace{n - 1 = 14}_{\text{category dividers}}$$
$$\underbrace{* * * \ \ldots \ * **}_{200} \ \underbrace{|||\ldots|||}_{14}$$
$$\binom{214}{14}$$

There are a total of $\binom{214}{14}$ different nut mixes we can make. That's a lot of possibilities in 200 kernels.

Collection 4

Pascal triangle, Binomial coefficients, Binomial theorem, anagrams...

1. (Saad) What is the coefficient of x^7y^3 in $(x+y)^{10}$? What about the coefficient of x^7y^2?

2. (Saad) How many anagrams exist for "SAAD MNEIMNEH" (we must keep the space, i.e. have two words)?

3. (Saad) How many words of length n can we make using the alphabet $\{a, b, c\}$. Solve this by first deciding how many positions in the word will be "a", then setting the rest using $\{b, c\}$. Finally use the addition rule and the Binomial theorem to get to the final answer.

4. (Soren) A person wants to create a tune of ten notes using two acoustic drums (for the sake of this problem we are not considering rhythm, only pitches). They are sitting in front of both these drums, with a low-pitched drum to be struck by their left hand, and a high-pitched drum to be struck by their right hand. Each drum can only produce one note, and for each of the ten notes this person has a choice of striking the low-pitched left drum or the high-pitched right drum.

   (a) Write this scenario in terms of a binomial expression, using your choice of variables (alongside a brief English definition of each variable).

   (b) How does this expression relate to the Pascal's triangle?

   (c) If we were to expand this expression, what would each term mean in English in terms of the problem? (i.e. "This term __ represents...")

   (d) How many distinct tunes can this person make?

5. (Vlad) Let's make a Pascal's pyramid. For a Pascal triangle, given a row consisting of: $\binom{n}{0}\binom{n}{1}...\binom{n}{n}$, let n be the row's height. Think of rows as "slices" of the triangle. For a Pascal's pyramid, a slice at height n is a triangle, stacking these triangles on top of each other forms the pyramid. For a height of n, a slice will look like this:

$$
\begin{array}{cccccc}
 & & & & & \binom{n}{0,n,0} \\
 & & & & & \vdots \\
 & & & \binom{n}{0,3,n-3} & \cdots & \binom{n}{n-3,3,0} \\
 & & \binom{n}{0,2,n-2} & \binom{n}{1,2,n-3} & \cdots & \binom{n}{n-2,2,0} \\
 & \binom{n}{0,1,n-1} & \binom{n}{1,1,n-2} & \binom{n}{2,1,n-3} & \cdots & \binom{n}{n-1,1,0} \\
\binom{n}{0,0,n} & \binom{n}{1,0,n-1} & \binom{n}{2,0,n-2} & \binom{n}{3,0,n-3} & \cdots & \binom{n}{n,0,0}
\end{array}
$$

Where $\binom{n}{x,y,z} = \frac{n!}{x!y!z!}$ and $x + y + z = n$

For this example let a generic trinomial be labeled as $\binom{n}{x,y,z}$. Notice the following patterns: As you go to the right on the slice, x increases by 1 for every step, while z decreases by 1, and as you go up, x decreases by one and y increases by one and z stays the same.

(a) What is the sum of trinomial coefficients at height n?

(b) Make a statement about the sides of the triangle for each slice. Why does this occur?

(c) Find a rule for generating the pascal's pyramid slice at height n given a slice at height $n-1$. What is the intuition behind this rule?

6. (Zach) State the binomial theorem. What are the terms in the sum $(2+3)^5$? What is the sum equal to?

7. (Tasmina) It's prom night and everyone has a choice of a corsage. There are 10 white corsages, 15 red ones, and 20 blue ones.

   (a) Let's say there's enough people to account for all corsages. How many different combinations of people with corsages are there?

   (b) Now let's say there's only 10 people. Now how many ways are there to choose different combinations of people with corsages?

8. (John) Consider a square pyramid where each of the four triangular faces is a pascal's triangle (up to height 10).

Solutions

1. The coefficient for x^7y^3 is $\binom{10}{3}$ by the Binomial theorem. The coefficient for x^7y^2 is 0 because x^7y^2 is not a term of the binomial $(x+y)^{10}$.

2. There are 13 characters in total (including the white space). In addition, "A" repeats twice, "M" repeats twice, "N" repeats twice, and "E" repeats twice. So the number of anagrams is $13!/(2!2!2!2!)$. However, this will include those anagrams which start or end with space, which we do not want. So we must subtract $2 \times 12!/(2!2!2!2!)$. So we obtain $(13-2)12!/(2!2!2!2!)$.

   Another approach is to simply find the anagrams of "SAADMNEIMNEH" and then slice it in two parts by choosing a position out of the 11 positions for space. So we get $11 \times 12!/(2!2!2!2!)$.

3. The number of positions that can be "a", let's call it k, can vary from 0 to n. There are $\binom{n}{k}$ ways for choosing those positions. The rest of the positions can be set using $\{b, c\}$ in 2^{n-k} ways. Using the addition rule, we have

   $$\binom{n}{0}2^{n-0} + \binom{n}{1}2^{n-1} + \ldots + \binom{n}{n}2^{n-n}$$

   By the Binomial theorem, this is $(2+1)^n = 3^n$, which is the expected answer.

4. An important property of this problem to observe is an implicit assumption of the nature of music: different orders of notes result in different tunes. This means every tune of ten notes is unique, unless it recreates the exact order of selection. Hopefully this was obvious (in the same way different orders of letters creating different words is hopefully obvious).

   (a) There are 10 decisions, and 2 choices for each decision. Call the left drum L, and the right drum R:

   $$(L+R)^{10}$$

   (b) The coefficients of the expanded above expression would exactly match the 10th row of the Pascal's Triangle:

   1 10 45 120 210 252 210 120 45 10 1

(Reproducing this as shown isn't expected; I do this for visual's sake. What *is* important is to recognize the Pascal's Triangle where you see it, because it allows one to borrow its properties: alternating sums, symmetry, row sums being powers of 2, etc.)

(c) For, say the $210L^4R^6$ term, we could say: "The term $210L^4R^6$ represents 210 tunes that have 4 left strikes and 6 right strikes." It is important to observe that, in this problem, the binomial coefficients are referring to the total number of tunes with a specific number of Ls and Rs, though each tune is unique (as mentioned earlier).

(d) 2^{10} distinct tunes. We can reach this expression a variety of ways: observing 10 decisions of 2 choices, the sum of the 10th row of the Pascals' triangle, etc.

5. (a) The sum of all trinomial coefficients is the sum of all the ways to make a string with n characters where every character has 3 options. So the total should be equal to 3^n.

(b) Each side of the triangle looks like a row of Pascal's triangle at height n. This happens because each side has one of the coefficients x, y, z equal to 0. So one of the coefficients in the denominator will be 0!. Assume we're looking at the side where $z = 0$, this implies that $y + z = n$ which implies $y = n - z$ so $\binom{n}{0,y,z} = \frac{n!}{y!z!} = \frac{n!}{z!(n-z)!} = \binom{n}{z} = \binom{n}{y}$

(c) $\binom{n}{x,y,z} = \binom{n-1}{x-1,y,z} + \binom{n-1}{x,y-1,z} + \binom{n-1}{x,y,z-1}$ where if any of the terms x, y, z are less than 0 the expression $\binom{n}{x,y,z} = 0$. The intuition behind this expression is as follows: Tof ind the number of ways to make an anagram with x xs, y ys and z zs we could:

i. add an x to an anagram with $x - 1$ xs y ys and z zs

ii. add a y to an anagram with x xs $y - 1$ ys and z zs

iii. add a z to an anagram with x xs y ys and $z - 1$ zs

These cases give disjoint sets so by addition rule we can add them to get our answer.

6. The binomial theorem is: $(x+y)^k = \binom{k}{0}x^0y^k + \binom{k}{1}x^1y^{k-1} + ... + \binom{k}{k}x^ky^0$. So, we have: $(2+3)^5 = \binom{5}{0}2^03^5 + \binom{5}{1}2^13^4 + \binom{5}{2}2^23^3 + \binom{5}{3}2^33^2 + \binom{5}{4}2^43^1 + \binom{5}{5}2^53^0 = 2^03^5 + 5 * 2^13^4 + 10 * 2^23^3 + 10 * 2^33^2 + 5 * 2^43^1 + 2^53^0 = 243 + 810 + 1080 + 720 + 240 + 32 = 3125$

7. (a) This is a simple anagrams problem. In total, there are 10+15+20 corsages, or 45. White corsages repeat 10 times, red ones 15 times, and blue ones 20 times. Thus, the number of anagrams is 45!/(10!15!20!).

   (b) This may seem difficult at first but think about the number of possibilities for the people. Since there are 10 people and the least number of a single corsage is 10 (10 white corsages), that means all the people have equal probability of wearing any of the three corsages. Thus, this becomes a simple integer solutions problem, where we have a box of white corsages (w), a box of red corsages (r), and a box of blue corsages (b), all of which add up to 10 corsages in total. These can be represented as $w + r + b = 10$. Since the conditions for each is already set to be greater than or equal to zero (since we don't require anybody to have a white corsage), this becomes $\binom{10+3-1}{3-1}=\binom{12}{2}=66$.