CSCI 150 Spring 2025 TA's questions

Tasmina, Vladislav, John,
Nicholas, Zach, Zhen Tao,
Nathaniel, Femi, Soren
with help and edits by Saad
Computer Science, Hunter College

Collection 5

Topics: Propositions, truth tables, AND, OR, NOT operators, Implication, if and only if (iff), contrapositive, Boolean functions.

1. (Zach) When is the following statement true: If the moon is made of cheese, then every student in Discrete Structures will earn a perfect score on the first midterm. Explain your reasoning.

2. (Tasmina) It's spring break and you're deciding how to spend it. Your mom is telling you to stay home and study, but your friends have been begging you to go on a trip with them. If you do go, you can go to Hawaii or Singapore, but must avoid London because it's too rainy there. Write out all the possible ways you can spend your spring break in terms of AND, OR, or NOT (or using multiple of them at once).

3. (Vlad) Consider the statement: "If the cat is curious, then if a cat is well-behaved, then the cat will not bite." Is this statement logically equivalent to: "If the cat is curious and well-behaved, then the cat will not bite."? Let A=cat is curious, B=cat is well-behaved, C=cat will bite. Rewrite both statements in the logic notation. Prove your answer to the first question in two ways:

   (a) Make a truth table.

   (b) Use the identity $P \Rightarrow Q = \neg P \vee Q$ and manipulate the logical expression until you can show that the expressions are equivalent or not equivalent.

4. (Nathaniel) Here is a excerpt from Kipling's poem "If":

   If you can talk with crowds and keep your virtue,
   Or walk with Kings—nor lose the common touch,
   If neither foes nor loving friends can hurt you,
   If all men count with you, but none too much;
   If you can fill the unforgiving minute
   With sixty seconds' worth of distance run,
   Yours is the Earth and everything that's in it,
   And—which is more—you'll be a Man, my son!

   Write this statement as a series of implications in proper notation, then write their contrapositives.

5. (Soren) Write the contrapositives of the following statements:

   (a) $x > 3 \quad \Rightarrow \quad x \in S \vee x \in T$
   (b) $f(x)$ is an odd function $\quad \Rightarrow \quad$ $f(x)$ is one-to-one
   (c) The sky is blue $\quad \Rightarrow \quad$ The sun is out $\quad \Rightarrow \quad$ It is day

6. (John) We know that a rectangle is not a square but a square is a rectangle.

   (a) Express the reason why using propositional logic and/or truth tables.
   (b) Is this relationship biconditional (iff)?

Solutions

1. The statement is always true because the hypothesis is always false. Recall that when we have an implication and the hypothesis is false, the implication is true regardless of the truth value of the conclusion. In this situation, we say that the statement is vacuously true.

2. There are multiple ways to interpret this problem, depending on what you think to include. The following includes every possible action: (((Stay at home) AND (Study)) OR ((Go to Singapore) OR (Go to Hawaii))) AND (NOT (Go to London)). Note that NOT (Go to London) can be placed in multiple areas; this is just one solution.

3.

4. Let $A =$ "you can talk with crowds "

   $B =$ "[you can] keep your virtue"

   $C =$ "[you can] walk with Kings"

   $D =$ "[you] lose the common touch"

   $E =$ "foes [can hurt you]"

   $F =$ "loving friends can hurt you"

   $G =$ "all men count with you"

   $H =$ "[some men count] too much"

   $I =$ "you can fill the unforgiving minute With sixty seconds' worth of distance run"

   $K =$ "Yours is the Earth"

   $M =$ "[your is] everything that's in it"

   $N =$ "you'll be a Man, my son"

   Then we can write the full statement as

   $$A \wedge B \vee \neg(C \vee D) \Rightarrow K \wedge M \wedge N$$

   $$\neg(E \vee F) \Rightarrow K \wedge M \wedge N$$

   $$G \wedge \neg H \Rightarrow K \wedge M \wedge N$$

   $$I \Rightarrow K \wedge M \wedge N$$

   Apply DeMorgan's law to negate both sides, and switch the implication to get the contrapositive.

   $$\neg K \vee \neg M \vee \neg N \Rightarrow \neg A \vee \neg B \wedge (C \vee D)$$

$$\neg K \vee \neg M \vee \neg N \Rightarrow E \vee F$$

$$\neg K \vee \neg M \vee \neg N \Rightarrow \neg G \vee H$$

$$\neg K \vee \neg M \vee \neg N \Rightarrow \neg I$$

Finally we translate it back to text:

You cannot talk with crowds or keep your virtue,
And, walking with Kings, lose the common touch
Foes and loving friends can hurt you,
And not all men count with you, or some too much
You cannot fill the unforgiving minute
With sixty seconds' worth of distance run
If yours is not the Earth or everything that's in it;
Or —which is more—you'll not be a Man my son!

While logically equivalent to the original poem, this version is somewhat less inspiring.

5. (a) $x \notin S \wedge x \notin T \quad \Rightarrow \quad x \leq 3$
   Reached by applying DeMorgan's law, as well as observing the inverse of "greater than" being the combination of the other two comparative possiblities (less than or equal to).
   (b) $f(x)$ is not one-to-one $\quad \Rightarrow \quad$ $f(x)$ is not an odd function
   A function does not need to be either of the even/odd parities (in contrast to integers); so the negation of this antecedent (the "if" portion of an implication) is not "$f(x)$ is an even function".
   (c) It is night $\quad \Rightarrow \quad$ The sky is blue $\wedge$ The sun is set
   Note that this isn't the expected statement:
   (It is night $\quad \Rightarrow \quad$ The sun is set $\quad \Rightarrow \quad$ The sky is not blue)
   The first implication must be treated as a proposition to negate, and the contrapositive is conversed upon the second implication. This is because implication is not commutative; therefore only the aforementioned procedure works. A truth table is provided in Figure 1.

6. (a) Let:
   i. $S =$ is a square

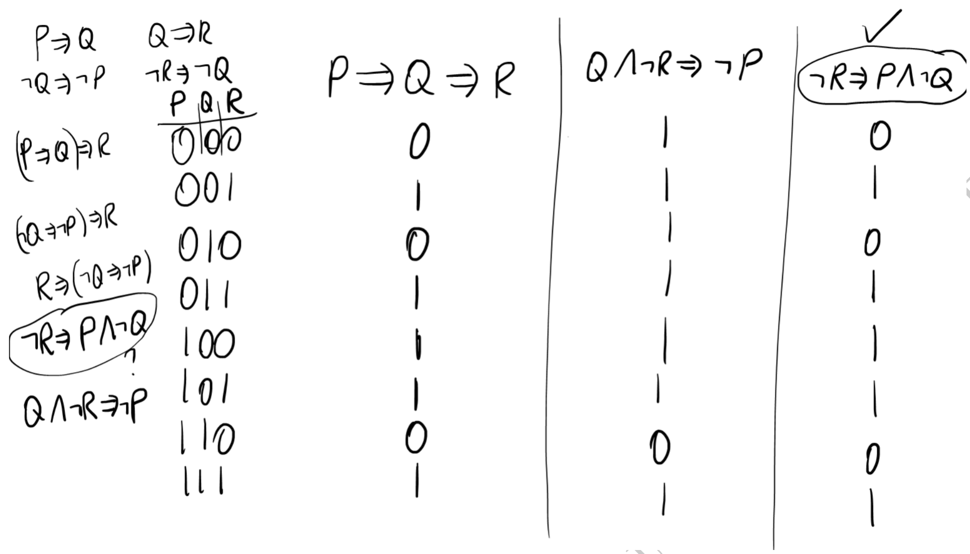

Figure 1: The truth table of a chained implication and attempting two contrapositives

ii. $R =$ is a rectangle
iii. $Q =$ is a quadrilateral with 4 right angles
iv. $E =$ has equal sides

We can then establish the following statements:
$S \iff Q \wedge E$
$R \iff Q$
We can immediately tell the two are not equal if E is false and Q is true.

(b) Let:

i. $S =$ is a square
ii. $R =$ is a rectangle

Since we know $S \Rightarrow R$ is true but $R \Rightarrow S$ is false, this statement is not biconditional.

Collection 6

Topics: Proofs. Direct proofs, proofs by contradiction, contrapositive, if-and-only-if, parity-based proofs, existential proofs, etc...

1. (Zhen) **The Case of the Missing Cookie**
   Today is a sunny Tuesday day, and Detective Chanel has just received a case from someone regarding a missing cookie. Upon arrival at the scene, she conducted some investigation and was able to find 3 suspects: Abbey, Bob, and Cherry. However, Chanel can't seem to pinpoint who exactly ate the cookie! Feeling stuck, Chanel asked you for help. You, being the magical person that you are, used your magic carrot wand which provided you with the following insights:

   (a) If Abbey didn't go to the park, then Abbey ate the cookies

   (b) If Cherry ate the cookie, then today is not Tuesday

   (c) If it's sunny, then Abbey went to the park and Bob didn't play games

   (d) If Cherry and Bob didn't eat the cookie, then Abbey didn't go to the park or Cherry did go to the library

   (e) If Cherry didn't eat the cookie and Cherry go to the library, then Bob is playing games

   These insights are all true, and they'll help you solve this thrilling mystery. Time for you to get rolling: Who ate the cookie?

2. (Zhen) **Organic Chemistry**
   The general formula for alkanes, a class of hydrocarbon, can be given by:

$$\mathrm{C}_n\mathrm{H}_{2n+2}$$

   where the subscript denotes the count of each elements, and $n \geq 1$. Prove that for any given alkane, the sum of its elements has the same parity as the number of C (carbon).

3. (Zach) Prove there is a bijection between the cartesian product $\mathbb{Z} \times \mathbb{Z}$ and the cartesian product $\mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$. Which proof technique did you use?

4. (John) The newly appointed director of Hogwarts wants to design a self-building magic staircase that gives access to every one of the four houses (Gryffindor, Slytherin, etc.). After taking measurements to accommodate for this new structure, he discovered the following:

   (a) Each floor of Hogwarts is positioned an even number of feet above the ground (i.e. the first floor is 0 feet above the ground while the second floor is 50 feet above the ground).

   (b) The entrance of each house is a large archway where both the start of the archway is positioned an odd number of feet relative to the leftmost wall of the floor and the width of the entrance is a multiple of 4 feet (if we imagine these locations to be part of a coordinate system, the archway on the second floor would start at i.e. (17, 50)).

   (c) Finally, the entrance of each house is some even number of feet from the front-most wall of the floor (once again using the coordinate system, the archway on the second floor would start at i.e. (17, 50, 18)).

   Each step of the staircase is 1 cubic foot. If the director wants to build the staircase such that the first step is positioned at (0, 0, 0) and its final step is perfectly aligned to the center of each house's archway, prove that the number of steps required to reach the entrance of a house is odd.

5. (Tasmina) Prove that $0.\overline{999}$ is equal to 1.

6. (Soren) Obtain an image of the NYC subway system (for copyright caution, this image is not provided; though any obtained image is surely consistent for this problem). Prove that, in order to travel from the Bronx to Queens via subway alone, at least one subway transfer between two subway lines is required. In this proof, state the proposition you are proving as the technique used (if you are using one; there are several solutions/methods possible).

7. (Nathaniel) Prove by contradiction that there are an infinite number of triangular numbers.

Solutions

1. It is crucial to observe that today is a sunny Tuesday day (given in the first line of the problem) because those will help decode the insights:

    - The first thing to notice is that today is a Tuesday. If we take the contrapositive of (b), we get "If today is Tuesday, then Cherry did not eat the cookies". Thus Cherry did not eat the cookie
    - In addition, today is also sunny. Thus for (c), we're able to conclude that Abbey went to the park and Bob didn't play games. Now note that for (a), although the insight is true, we can't deduce whether Abbey ate the cookies or not, since p is false (false implies q will always be true, doesn't matter what q is)
    - Since we deduced that Bob didn't play games, we can take the contrapositive of (e) to get "If Bob didn't play games, then Cherry ate the cookie or Cherry didn't go to the library". Since we deduced from previous that Cherry didn't eat the cookie, that must mean that Cherry didn't go to the library.
    - Now that we know Cherry didn't go to the library and Abbey went to the park. The final step is taking the contrapositive of (d), which becomes "If Abbey went to the park and Cherry didn't go to the library, then Cherry or Bob ate the cookie. We have already concluded that Cherry didn't eat the cookie, thus Bob has to be the one.

    Therefore, Bob ate the cookie!

2.

3. The question is basically asking you to prove that the cardinality of the two cartesian products is the same. We just need to find a function that is a bijection between the two sets. Consider the bijection f: $\mathbb{Z} \rightarrow \mathbb{N} \cup \{0\}$, which maps each integer to a natural number defined by: f(x) = { 2x if $x \geq 0$; -2(x)-1 if $x < 0$ }. In this way, the negative integers get mapped to the odd natural numbers and the positive integers get mapped to the even natural numbers, and 0 gets mapped to 0. Given an ordered pair of integers, we use f on each integer to compute its associated natural number. The result is a pair of natural numbers. Note that an arbitrary pair of natural numbers has a unique pair of integers that gets mapped to it because f is a bijection, so our resulting function is one-to-one. (If we change one of the integers in the pair of

integers, the pair of natural numbers that it gets sent to is different). Our function is also onto because given any pair of natural numbers, by the definition of f there is guaranteed to be an integer that gets sent to every natural number, so there is always a pair of integers that gets sent to a pair of natural numbers. Hence, there is a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$. The proof technique used in this example is proof by existence. We just come up with an example to prove the statement.

4. Let us count the number of stairs necessary for each dimension of the staircase.

   (a) For the height, we know the staircase starts at y=0 and ends at some even number of feet above the ground. This means that height $h = 2k, k \in \mathbb{Z}$, and since we know that each step is 1 foot tall, it will require h number of stairs to reach the floor of a house's entrance.

   (b) For the width, we know the start of the archway is positioned an odd number of feet from the leftmost wall. Let us say that the start position $s = 2m + 1$ and the length $d = e - s = 4n$ where $e, n, m \in \mathbb{N} \wedge e > s$. Since we want the staircase to be aligned to the center of the archway, we must find its midpoint. To find the midpoint, do the following:

   $$s + \frac{d}{2} = (2m + 1) + \frac{4n}{2} = 2m + 2n + 1 = 2(m + n) + 1$$

   Thus, we know the number of stairs to reach the center of the archway is odd.

   (c) For the length, follow a similar procedure to part a. We know the length is some even number of feet relative to the starting point, meaning the length $l = 2b, b \in \mathbb{Z}$; we will require l stairs to reach this position.

To calculate the total number of stairs of 1 cubic foot required to reach the entrance, we can add the number of steps required to reach each individual component of the vector representing the entrance's location. Doing this will give you

$$(2k) + (2(m + n) + 1) + (2w) = 2(k + m + n + w) + 1, k, m, n, w \in \mathbb{Z}$$

meaning that the total number of steps must be odd.

5. This is a direct proof. Iff we manipulate the decimal $0.\overline{999}$, we get the following:

$$N = 0.999...$$
$$10N = 9.999...$$
$$10N - N = 9.999... - 0.999...$$
$$9N = 9$$
$$N = 9/9$$
$$N = 1$$

This shows that $0.\overline{999}$ is equal to 1.

6. We can word the proposition to be proved as such: "If one travels from the Bronx to Queens using the subway alone, then they perform a subway transfer".

   One method of proving this is by proving the contrapositive instead: "If one does not perform a subway transfer, then one does not travel from the Bronx to Queens using the subway alone". We can then prove this via direct proof: assuming the antecedent (first clause) to be true, observe truthful implications until the conclusion (second clause) is reached. Because the possible cases of a true conclusion are small (the number of subway lines we can choose from is an easily countable number), we can prove the conclusion by exhausting all cases by observing which boroughs all subway lines from the Bronx travel:

   B, D, 2, 4, 5 Bronx → Manhattan → Brooklyn

   1, 6 Bronx → Manhattan

   We covered all cases (subway lines). None of them travel to Queens. Therefore, without a subway transfer, we cannot travel from the Bronx to Queens via subway alone.

7. To set up our contradiction, we will assume that there are finite triangular numbers, and that some number t is the largest one. t can be represented as an equilateral triangular arrangement of t dots with finite side length n. Since n is finite, we can add another row of $n + 1$ dots, making a larger equilateral triangle. The new total number of dots is a triangular number that's larger than t; therefore t is not the largest trangular number. This contradiction shows that no t can exist.

Collection 7

Topics: Size of infinite sets. Countable vs uncountable. Techniques for proving countable: Using set operations on countable sets, bijection with $\mathbb{N}$, or listing elements in such a way that each will have a finite rank. Techniques for proving uncountable: Bijection with an uncountable set, Cantor's diagonal proof.

1. (Vlad) In class we've shown that positive rational numbers are countable using the fact that every rational number could be represented by a tuple (a, b) where a and b are positive integers. Suppose that there is a set of magic numbers M such that each magic number corresponds to a unique tuple (a, b, c) where a, b, c are positive integers and vice versa (there is a bijection between the set of magic numbers and the set of (a, b, c) tuples). Try to outline the proof that magic numbers are countable by using the following steps (the proof has a lot of similarities to the proof for tuples (a, b)):

   (a) In the proof that rational numbers are countable we've ordered the rational numbers in a table where the rows represent the value of a and columns represent the value of b. How can we arrange the set of (a, b, c) tuples in a similar way?

   (b) In the proof that rational numbers are countable we've used a diagonal line going through a_j, b_j to bound the numbers in a triangular region. The line passes through all (a, b) tuples where $a + b = k$ where k is some integer. What geometric object would we need to use to bound the tuples in our new proof? What sort of tuples will this object intersect?

   (c) In the proof for (a, b) tuples we were able to bound the rank of a rational number $\frac{c}{d}$ by adding up all tuples that sum up to $c + d$, resulting in a total of $1+2+3+4+5...+(c+d) = (c+d)(c+d+1)/2$. How does this carry over to our case here.

2. (Zach) You are given a 2D grid whose rows and columns are infinite in both directions. Prove that the cells in the grid are countable.

3. (Tasmina) You've been teleported to the infamous Library of Babel, which has an infinite number of books. To get out, you must establish an order of the books such that they are countable. Will you get out? If so, how? If not, why? (*Hint*: You can categorize books by their titles.)

4. (John) Given a graph with infinite vertices, is the collection of all possible graphs that can be formed from those vertices countable?

5. (Soren) These are a series of questions to help better understand the nature of countability. Therefore, this is written with the intention of conceptual exploration and catalyzing understanding rather than striving for correct responses.

   (a) Determine whether these sets are countable or uncountable, with a brief explanation for your reasoning:

      i. The set of all water molecules in the ocean

      ii. The set of all possible orientations for a boat in the ocean $f(x, y, r)$; x and y referring to latitude and longitude coordinates, r referring to rotation in degrees

      iii. The amount of time until the heat death of the universe

      iv. The set of all instances of time until the heat death of the universe

      v. What do the answers to the four questions above tell us about the nature of countability as it relates to the physical world?

   (b) Are you able to describe the exact quantity of π using a finite numerical description without using the symbol/notation itself? $\sqrt{2}$? What about $0.\overline{123456789}$ (Hint: search up "repeating decimal calculator"). What does this mean for the ability to sort these two numbers?

   (c) In how many dimensions is $\mathbb{R}$ infinite (looking at an example Cantor's diagonal bijection could be helpful)? $\mathbb{N}$? The set of finite subsets of $\mathbb{N}$? The set of infinite subsets of $\mathbb{N}$?

Solutions

1. To be filled out by Vlad...

2. Pick any cell in the 2D grid. That's our starting point. Label this cell 1. Move right one, and label that cell 2. Next, move up one, and label this cell 3. Then, from there, move left one, label this cell 4 (this cell is directly above the starting point). Next, move left one again, and label this cell 5. From there, move down one, and label this cell 6. Move down yet again, labeling the cell 7. Notice that we have just traveled in a rectangle. To label the remaining cells from where we left off, form continuously larger rectangles with each side one cell longer than the corresponding side of the adjacent inner rectangle. In this way, we account for all of the cells, and thus the cells in the infinite 2D grid are countable because each cell has a unique natural number that maps to it.

3. You will get out! The trick is to identify that each book is defined by their title, which is a finite sequence of letters. Thus, we can organize the books by the length of their titles, such that all 1-letter titles come first, followed by 2-letter titles, and so on. Within each length, the books will further be arranged alphabetically, allowing us to identify the exact rank of any given book, thus making the set countable.

4. ...

5. (a)
   i. An absurdly unimaginable quantity, yet still finite; therefore countable. Only countable theoretically, of course; but math does not differentiate between theoretically possible and practically possible. That's a question for physics.
   ii. Each of x, y, r maps to a finite range of real numbers. As explored in HW8 #1, each finite range real numbers is as large as the set of real numbers themselves. Therefore, this is uncountable.
   iii. Also an absurdly unimaginable quantity, yet still finite; therefore countable.
   iv. An instance of time can be interpretted as any real value of t. Also mapping to real numbers, this quantity is uncountable.
   v. Anything physically representable is countable. It is only when something is mapped onto the real numbers, meaning it is conceptually abstracted into a model that allows for

any possible number, that something becomes uncountable. Infinity, without convergence (i.e. the limit of something infinite being finite), is not a concept that manifests physically.

(b) All irrational numbers cannot be expressed as a finite quantity; instead we come up with a concept for a number and invent new notation to express the value of this number, to be only approximated numerically when not represented in terms of this concept (i.e. $\sqrt{2}$ is the number that equals 2 when squared, π is the ratio of a circle's circumference to its diameter; these quantities capture infinity itself in its complete disagreement with numbers). Because of this, we cannot properly determine their exact order within all possible irrational numbers.

$0.\overline{123456789}$, on the other hand, is finitely representable as $\frac{13717421}{111111111}$; so we can determine an order for even a number like this. In fact, all repeating decimal numbers are actually finite; the repetitions are merely disagreement with the expressed base (in our case, base 10). For example, $\frac{1}{3}$ in base 10 is $0.\overline{333}$, whereas in base 3 it is 0.1 .

(c) $\mathbb{R}$ is "multi-dimensional infinite" (I am making this term up; there's probably a formal way of saying this): both the numbers themselves are infinite as well as the collection of numbers. This is intuitive looking at example bijections for in Cantor's diagonalization proof: infinity expands both downwards and to the right ("two dimensionally infinite").

Meanwhile, $\mathbb{N}$ is "single-dimension infinite": it is only infinite in terms of the collection itself; the terms themselves are finite. Because of this, we cannot represent a "two-dimensional infinity" in terms of a "one-dimensional infinity"; only other provably "one-dimensionally infinite" sets can be expressed in terms of a bijection with $\mathbb{N}$.

A similar argument, alongside intuition explored in part (b), can be made for infinite subsets of $\mathbb{N}$ being "multi-dimensionally infinite" and finite subsets of $\mathbb{N}$ being "singularly dimensionally finite".

Collection 8

Topics: Inclusion-Exclusion, Pigeonhole, proof by induction.

Collection 9

Topics: Recurrences.