CSCI 150 Discrete Mathematics
Test 1

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Thu. Oct. 10, 2024

Name: **EmplID**:

Number of anagrams you can make from your first name: Don't use space below

Recitation instructor (circle one): Shayan Anthony Taha P1:

Recitation day (circle one): Mon Wed Thu P2:

P3:

P4:

Write your name on this page.

Don't turn THIS page until it's time.

There are 10 pages (including this one).

There are 4 problems (with several parts).

Scrap paper will be provided.

Make sure no one sits in a seat next to yours.

Turn all your cell phones off and place them away (and no calculators).

If you need to leave (e.g. bathroom break), please give your test and all your cell phones to a proctor.

There is a total of 22 points, but the test will be graded over 20.

Write your answers neatly and clearly. Do no squeeze your answers between questions, use the dedicated space for each problem. Make sure everything is legible.

FYI: I tried my best to design questions that (1) cover most of the concepts we have seen, (2) mimic several ideas in recitations, homework, and sample test questions, and (3) present non-trivial but reasonable problems.

Problem 1: Seating people

(a) **(2 points)** In how many ways can we seat 20 people on 20 chairs if the chairs are in a straight line?

(b) **(2 points)** Assume that people have been seated already and that seating has been fixed. In how many ways can we choose 3 people if none of them are neighbors (sitting next to each other). *Hint*: Every choice of 3 people makes 4 consecutive groups out of the remaining ones, as shown below.

$$\underbrace{\circ\ \circ\ \circ}\bullet\overbrace{\circ\ \circ\ \circ\ \circ\ \circ}\bullet\overbrace{\circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ}\bullet\underbrace{\circ\ \circ}$$

[Use this page and the next page (both front and back) to answer Problem 1]

Answers to Problem 1 (here and back)

Problem 2: Trapezoidal numbers

A *trapezoidal* number is the sum of at least two consecutive positive integers. For instance, $3+4+5=12$ and $7+8=15$ are both trapezoidal numbers.

(a) **(2 points)** Let the trapezoidal number T_k^ℓ consist of adding the first ℓ integers starting with k. Express T_k^ℓ using a Σ notation, and evaluate the sum. The final expression should be in terms of k and ℓ.

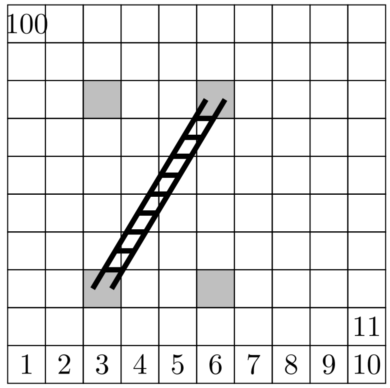
(b) **(2 points)** For a given integer $n \geq 2$, consider the set $S = \{1, 2, \ldots, n\}$. In how many ways can we make a trapezoidal number by summing consecutive integers in S? Explain your answer.

[Use this page and the next page (both front and back) to answer Problem 2]

Answers to Problem 2 (here and back)

Problem 3: Positive ladders

Consider a standard 10×10 snakes and ladders board.



In addition, this version of the game is unique in that:

- there are no snakes (yay!),
- there is only one ladder,
- the ladder must have a positive finite slope, as shown in the example above.

(a) **(2 points)** As you might have guessed, we would like to count the number of ways we can place a ladder with positive (and finite) slope. Use the the following procedure to guide you (also illustrated above), then apply the product rule.

```
                                                                                #ways
1. choose a square, call it A .............................................
2. choose another square B in the same row as A ...........................
3. choose another square C in the same column as A ........................
4. choose the square D that makes a rectangle with the three above ........
5. use the bottom left and upper right squares in (A, B, C, D) to draw the ladder .....   1

                                                                                -------
```

(b) **(2 points)** If the 4 chosen squares are (A, B, C, D), how many permutations of these choices, consistent with the procedure above, lead to the same ladder? Explain, adjust for overcounting, and report your final answer. *Hint*: Not all permutations of the four squares can be obtained by the procedure above. In other words, we are not overcounting by 4!.

(c) **(2 points)** Consider the following function:

$$f : \{1, 2, \ldots, 100\} \to \{2, 3, \ldots, 100\}$$

$$f(x) = \begin{cases} x & \text{if } x = 100 \\ y & \text{if } x \text{ is the bottom of the ladder and } y \text{ its top} \\ x + 1 & \text{otherwise} \end{cases}$$

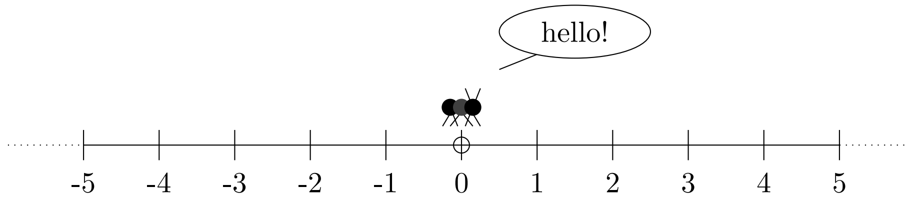
Is f one-to-one? Is it onto? Explain your answers.

[Use this page and the next page (both front and back) to answer Problem 3]

Answers to Problem 3 (here and back)

Problem 4: The lost ant

An ant stands at position 0 on the infinite line.



The ant will make a sequence of 100 moves. Each move consists of making one step to the left, or one step to the right, so the ant stays on the line at all times. If after making a sequence of 100 moves, the ant is positioned at 0, we say that the ant is not lost. Otherwise, the ant is lost. Therefore, a sequence of moves is *lossless* when the ant is not lost, or *lossy* otherwise.

(a) **(2 points)** How many lossless sequences are there? Explain how you arrive at your answer.

(b) **(2 points)** Explain how you can use the addition rule to obtain the number of lossy sequences (and find that number).

(c) **(2 points)** If the ant will only make an even number of left moves among the 100 moves, how many possible sequences are there? *Hint*: How many possible sequences are there if the ant makes exactly k left moves? Then use the addition rule and simplify your answer.

(d) **(2 points)** Assume now that if the ant makes a left move, it will either deposit pheromones or not. How many possible sequences are there, including pheromone pattern? *Hint*: same hint as above.

[Use this page and the next page (both front and back) to answer Problem 4]

Answers to Problem 4 (here and back)

[Scrap, will not be graded]

Solution

Problem 1: Seating people

(a) **(2 points)** In how many ways can we seat 20 people on 20 chairs if the chairs are in a straight line?

Using abstraction, this is a permutation, so there are 20! ways of seating 20 people on 20 chairs. Another way to approach this is by applying the product rule from scratch. There are 20 people to choose from for the first chair, 19 people to choose from for the second chair, etc... The result is $20 \times 19 \times \ldots \times 1 = 20!$. There is no overcounting because any permutation of the choices will change the seating. In other words, it is impossible for the same seating to correspond to different choices.

Grading: 2 points will be given to the correct answer. There is no room for partial credit here, but exception may apply.

(b) **(2 points)** Assume that people have been seated already and that seating has been fixed. In how many ways can we choose 3 people if none of them are neighbors (sitting next to each other). *Hint*: Every choice of 3 people makes 4 consecutive groups out of the remaining ones, as shown below.

$$\underbrace{\circ\ \circ\ \circ}\bullet\overbrace{\circ\ \circ\ \circ\ \circ\ \circ}\bullet\overbrace{\circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ}\bullet\underbrace{\circ\ \circ}$$

We need to count the number of ways we can make these 4 consecutive groups of people. Let x_1, x_2, x_3, x_4 be the sizes of the consecutive groups of people. We know that $x_1 + x_2 + x_3 + x_5 = 17$. To ensure the condition about neighbors, we must have $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 1$, and $x_4 \geq 0$. Therefore, we can write this as:

$$x_1 + (1 + x_2') + (1 + x_3') + x_4 = 17$$

$$x_1 + x_2' + x_3' + x_4 = 15$$

where all variables are ≥ 0. This has the standard solution $\binom{15+4-1}{4-1} = \binom{15+4-1}{15} = \binom{18}{3} = \binom{18}{15}$.

Grading: 1 point will be given for the approach, and 1 point for the correct answer. If everything is fine, but there is a slight mistake; for instance, in the formula itself, only 1/2 a point is taken away.

[Use this page and the next page (both front and back) to answer Problem 1]

Problem 2: Trapezoidal numbers

A *trapezoidal* number is the sum of at least two consecutive positive integers. For instance, $3+4+5=12$ and $7+8=15$ are both trapezoidal numbers.

(a) **(2 points)** Let the trapezoidal number T_k^ℓ consist of adding the first ℓ integers starting with k. Express T_k^ℓ using a Σ notation, and evaluate the sum. The final expression should be in terms of k and ℓ.

There are many ways one could express this, so any answer that is equivalent to the correct solution is acceptable.

$$\sum_{i=k}^{k+\ell-1} i = \sum_{i=0}^{\ell-1}(k+i) = \sum_{i=1}^{\ell}(k+i-1)$$

The formula for $a+(a+1)+\ldots+b$ has been given in class. Here $a=k$ and $b=(k+\ell-1)$. The formula was

$$\frac{a+b}{2}(b-a+1) = \frac{k+k+\ell-1}{2}\ell = \frac{2k+\ell-1}{2}\ell$$

But there are many equivalent answers. For instance, one could do the following:

$$\sum_{i=0}^{\ell}(k+i) = \sum_{i=0}^{\ell-1}k + \sum_{i=0}^{\ell-1}i = k\ell + \binom{\ell}{2} = k\ell + \frac{\ell(\ell-1)}{2} = \frac{2k\ell+\ell(\ell-1)}{2} = \frac{2k+\ell-1}{2}\ell$$

Grading 1 point is given for a correct sum notation, and 1 point for a correct expression in terms of k and ℓ. If there is a slight mistake in calculating the expression, 1/2 a point is taken way.

(b) **(2 points)** For a given integer $n \geq 2$, consider the set $S=\{1,2,\ldots,n\}$. In how many ways can we make a trapezoidal number by summing consecutive integers in S? Explain your answer.

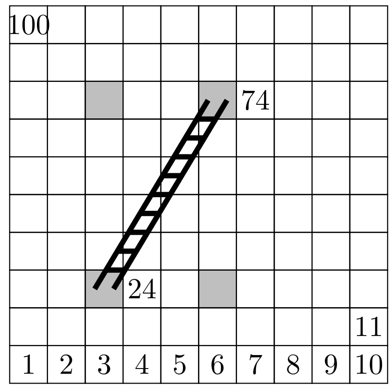
There are several ways of thinking about this. For instance, one could say that each trapezoidal number is determined by the smallest integer and the largest integer used in the sum. Therefore, a unordered pair $\{x,y\}$ of integers (observe that (x,y) and (y,x) represent the same trapezoidal number). There are $\binom{n}{2}$ such pairs. Another way is to identify that there are $n-1$ trapezoidal numbers that start with 1, and $n-2$ trapezoidal numbers that start with 2, etc... until just 1 trapezoidal number that starts with $n-1$. So we have $(n-1)+(n-2)+\ldots+1=\binom{n}{2}$ trapezoidal numbers, by the addition rule.

Grading: 2 points are given for the correct answer, but partial credit may be given based on the reasoning and the answer. There must be some explanation. The correct answer by itself will receive 1 point.

[Use this page and the next page (both front and back) to answer Problem 2]

Problem 3: Positive ladders

Consider a standard 10×10 snakes and ladders board.



In addition, this version of the game is unique in that:

- there are no snakes (yay!),
- there is only one ladder,
- the ladder must have a positive finite slope, as shown in the example above.

(a) **(2 points)** As you might have guessed, we would like to count the number of ways we can place a ladder with positive (and finite) slope. Use the the following procedure to guide you (also illustrated above), then apply the product rule.

```
                                                                                   #ways
1. choose a square, call it A .............................................................  100
2. choose another square B in the same row as A ......................................    9
3. choose another square C in the same column as A ...................................    9
4. choose the square D that makes a rectangle with the three above ...................    1
5. use the bottom left and upper right squares in (A, B, C, D) to draw the ladder .....    1

                                                                                   -------
                                                                                     8100
```

Grading: One possible strategy is to give 1/2 a point for each correct number put in the table. We will assume that the product is trivial to compute given the 4 numbers.

(b) **(2 points)** If the 4 chosen squares are (A, B, C, D), how many permutations of these choices, consistent with the procedure above, lead to the same ladder? Explain, adjust for overcounting, and report your final answer. *Hint*: Not all permutations of the four squares can be obtained by the procedure above. In other words, we are not overcounting by 4!.

There are only 4 permutations of (A, B, C, D) that are consistent with the procedure and are equivalent in terms of the ladder. There are the following:

$$(A, B, C, D), (B, A, C, D), (C, D, B, A), (D, C, A, B)$$

This is because the second square must be in the same row as the first, and the third square must be in the same column as the first. Therefore, we are overcounting by 4, and the final answer is $8100/4 = 2025$.

Grading: The important aspect here is to figure out that we are overcounting by 4, which deserves the two points. Partial credit may be given depending on the reasoning presented and the consistency of the approach with the reasoning.

(c) **(2 points)** Consider the following function:

$$f : \{1, 2, \ldots, 100\} \to \{2, 3, \ldots, 100\}$$

$$f(x) = \begin{cases} x & \text{if } x = 100 \\ y & \text{if } x \text{ is the bottom of the ladder and } y \text{ its top} \\ x+1 & \text{otherwise} \end{cases}$$

Is f one-to-one? Is it onto? Explain your answers.
The arguments below are essentially correct, but they lack some detail, which I omitted for clarity.

- The function is not one-to-one. For instance $f(99) = f(100) = 100$. Another example: assume $f(x) = y$ where x is the bottom of the ladder and y is the top of the ladder, and consider $y-1$. $f(y-1) = y$. Now we have $f(x) = f(y-1)$, but x and $y-1$ cannot be the same, since no ladder can go from $y-1$ to y; otherwise, it will be either horizontal or vertical. In the above picture $f(23) = f(74) = 75$.

  Partial credit: 1/2 a point for saying it's one-to-one because $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

- The function is also not onto. With x and y as described above, consider $x+1$. There is no z such that $f(z) = x+1$, because z would have to be x, but $f(x) = y \neq x+1$, again since the ladder cannot be horizontal or vertical. In the above picture, there is no z such that $f(z) = 24$.

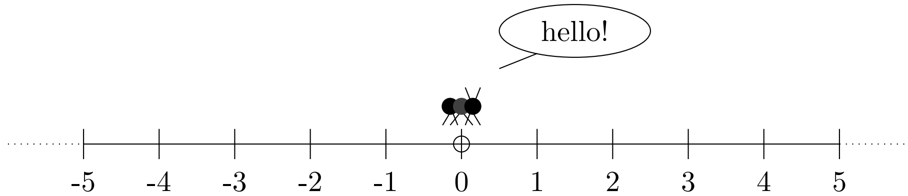
  Partial credit: 1/2 a point for saying it's onto because for any given $y \in \{2, \ldots, 100\}$, $x = y-1 \in \{1, \ldots, 100\}$ satisfies $f(x) = y$.

Grading: In each case, 1/2 a point is given for the correct answer, and 1/2 a point for some explanation that is reasonably correct.

[Use this page and the next page (both front and back) to answer Problem 3]

Problem 4: The lost ant

An ant stands at position 0 on the infinite line.



The ant will make a sequence of 100 moves. Each move consists of making one step to the left, or one step to the right, so the ant stays on the line at all times. If after making a sequence of 100 moves, the ant is positioned at 0, we say that the ant is not lost. Otherwise, the ant is lost. Therefore, a sequence of moves is *lossless* when the ant is not lost, or *lossy* otherwise.

(a) **(2 points)** How many lossless sequences are there? Explain how you arrive at your answer.

A sequence can be considered as a 100 bit pattern (e.g. Left=0, Right=1). A lossless sequence must have the same number of 0s and 1s. So how many binary patterns have 50 0s and 50 1s? That's $\binom{100}{50}$. One could also think of this as the number of anagrams that can be made from $\underbrace{L\ldots L}_{50}\underbrace{R\ldots R}_{50}$, which is $\frac{100!}{50!50!}$.

Grading 1 point for the correct answer, and 1 point for the reasoning. If the answer is consistent with the reasoning, but the reasoning is wrong or it does not capture the situation, then partial credit up to 1 point can be given.

(b) **(2 points)** Explain how you can use the addition rule to obtain the number of lossy sequences (and find that number).

By the addition rule, the number of lossless sequences and the number of lossy sequences must add up to the total which is 2^{100}. Therefore, the number of lossy sequences must be $2^{100} - \binom{100}{50}$.

Grading Again, 1 point for the correct answer, and 1 point for reasoning about the addition rule.

(c) **(2 points)** If the ant will only make an even number of left moves among the 100 moves, how many possible sequences are there? *Hint*: How many possible sequences are there if the ant makes exactly k left moves? Then use the addition rule and simplify your answer.

For a given k, there are $\binom{100}{k}$ sequences that make k left moves. By the addition rule, we have

$$\binom{100}{0} + \binom{100}{2} + \ldots + \binom{100}{100}$$

This is also the number of even subsets, which is $2^{100}/2 = 2^{99}$.

Grading: 1 point for setting up the answer, and 1 point for actually figuring out that it's half of 2^{100}.

(d) **(2 points)** Assume now that if the ant makes a left move, it will either deposit pheromones or not. How many possible sequences are there, including pheromone pattern? *Hint*: same hint as above.

Using the same idea, with the addition that a sequence with k lefts has 2^k ways of depositing pheromone, we have:

$$\binom{100}{0}2^0 + \binom{100}{1}2^1 + \ldots + \binom{100}{100}2^{100}$$

By the Binomial Theorem, this is $(1+2)^{100} = 3^{100}$.

Another way to see this is by observing that there are 3 types of moves, L, R, and L^*, where L^* is left with pheromone. So we have $3 \times 3 \ldots \times 3 = 3^{100}$.

Grading: 1 point for setting up the answer, and 1 point for actually figuring out the binomial theorem. If the correct answers was reached in some other way, that's ok too. Showing work and getting 3^{100} should receive full credit.

[Use this page and the next page (both front and back) to answer Problem 4]