CSCI 150 Discrete Mathematics

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Test 2, Thu. Nov. 14, 2024

Name **(2 points)**: Solution **EmplID** **(1 point)**: ********

What kind of socks does Saad wear? **(1 point)**: black and cotton (at the same time)

				This page:
Recitation instructor (circle one):	Shokri	Anthony	Taha	P1:
				P2:
Recitation day (circle one):	Mon	Wed	Thu	P3:
				P4:

Write your name on this page.

Don't turn *this* page until it's time.

There are 10 pages (including this one).

Scrap paper will be provided.

Make sure no one sits in a seat next to yours.

Turn all your cell phones off and place them away (and no calculators).

If you need to leave (e.g. bathroom break), please give me your test and all your cell phones.

There are 4 problems (with two parts each).

There are 4 points on this page (above), and each problem contributes 4 points (20 points in total). A 16/20 will be considered as a very good outcome, so you may focus on 3 problems.

Write your answers neatly and clearly. Do no squeeze your answers between questions, use the dedicated space for each problem. Make sure everything is legible.

FYI: I tried my best to design questions that (1) cover most of the concepts we have seen, (2) mimic several ideas in recitations, homework, and sample test questions, and (3) present non-trivial but reasonable problems.

Problem 1: Contradiction and contrapositive

(This question is inspired by the proof that primes are infinite presented in class, but has nothing to do with primes.)

(a) **(2 points)** We are given that a function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(x) > x$. Let $\mathbb{P}$ be a subset of $\mathbb{N}$ with the following mysterious property:

$$x \notin \mathbb{P} \Rightarrow f(x) \in \mathbb{P}$$

Prove that $\mathbb{P}$ is infinite using a proof by contradiction.

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(This question is designed to check your understanding of the contrapositive and negation, and how you handle the notion of even/odd.)

(b) **(2 points)** Consider the following statement:

$$xy - x + 1 \text{ is even} \Rightarrow x \text{ is even} \vee y \text{ is even}$$

State the contrapositive and prove it. *Hint*: Use DeMorgan's law.

Answers to Problem 1.

(a) In a proof by contradiction, we start with the negation of the statement we want to prove. We want to prove that $\mathbb{P}$ is infinite.

Assume that $\mathbb{P}$ is finite. Then $\mathbb{P} = \{x_1, x_2, \ldots, x_n\}$. Let $x \in \mathbb{N}$ be larger than every element of $\mathbb{P}$. Since $\mathbb{P}$ is finite, x can be constructed. For instance $x = 1 + \sum_{i=1}^{n} x_i$ (there are many possible ways to define x). Since $x > x_i$ for all $i = 1 \ldots n$, then $x \notin \mathbb{P}$. By the mysterious property, $f(x) \in \mathbb{P}$. But $f(x) > x > x_i$ for all $i = 1 \ldots n$, so $f(x) \notin \mathbb{P}$, a contradiction. Therefore, $\mathbb{P}$ is infinite.

(b) The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$:

$$x \text{ is odd} \wedge y \text{ is odd} \Rightarrow xy - x + 1 \text{ is odd}$$

Here, DeMorgan's law was used to negate Q, which is of the form $A \vee B$, to give $\neg A \wedge \neg B$.
Now if x and y are both odd, we can write: $x = 2k + 1$ and $y = 2k' + 1$. Therefore,

$$xy - x + 1 = (2k + 1)(2k' + 1) - (2k + 1) + 1 = 4kk' + 2k + 2k' + 1 - 2k - 1 + 1$$

$$= 4kk' + 2k' + 1 = 2(2kk' + k') + 1 = 2k'' + 1$$

Therefore, $xy - x + 1$ is odd.

Grading policy: For part (a), 1 point for starting correctly, another 1 for finishing. For part (b), 1 point for a correct contrapositive, another for proving it by reasoning about odds. Partial credit may be given when appropriate for both parts (a) and (b).

Problem 2: Other kinds of proofs...

(This question is a modification of a question presented in class.)

(a) **(2 points)** Show how you can use the principle of induction to prove the following statement for all integers $n \geq 0$:

$$\prod_{i=1}^{n}\left(1+\frac{2}{i}\right) = \binom{n+2}{2}$$

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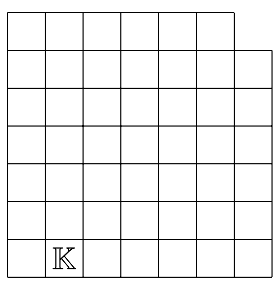
(This question is inspired by a question about domino tiling presented in class, but has nothing to do with dominos.)

(b) **(2 points)** In a modified game of chess, the board is 7×7 with a missing square as shown below. In addition, the knight moves by **3** squares in the vertical or horizontal direction, and **1** square in the other direction (as opposed to 2 and 1 for the regular knight). One concern is that the knight may land on the missing square. Prove that this can never happen given the knight's initial position, marked below by $\mathbb{K}$.



Hint: Use some parity based argument.

Answers to Problem 2.

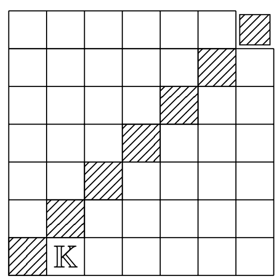
(a) Base case: When $n = 0$, we have $\prod_{i=1}^{0}(1 + 2/i) = 1 = \binom{0+2}{2}$ (we used the fact that the empty product is 1). So the property is true when $n = 0$.

For the inductive hypothesis, assume that the property is true for some $k \geq 0$, namely $\prod_{i=1}^{k}(1+2/i) = \binom{k+2}{2}$.

Inductive step: We want to show that for all $k \geq 0$, $P(k) \Rightarrow P(k+1)$. Consider $k+1$:

$$\prod_{i=1}^{k+1}\left(1+\frac{2}{i}\right) = \prod_{i=1}^{k}\left(1+\frac{2}{i}\right)\cdot\left(1+\frac{2}{k+1}\right) = \binom{k+2}{2}\frac{k+3}{k+1}$$
$$= \frac{(k+2)(k+1)}{2}\frac{k+3}{k+1} = \frac{(k+3)(k+2)}{2} = \binom{k+3}{2} = \binom{(k+1)+2}{2}$$

(b) We can define even and odd squares by mimicking the colors of a chessboard. For instance, as shown below, all squares on the diagonal (including the missing square) have the same parity, which is opposite to that of the kinght's initial square (not all shadings are shown).



The knight moves by 4 squares each time; therefore, it will always land on a square with the same parity as its original square. If the knight ever reaches the missing square, this means that it landed on the opposite parity, which cannot happen.

Grading policy: Since both components of a proof by induction are important, 1 point is given to the base case, and 1 point to the inductive step. There must be enough work in the inductive step to claim a full 1 point.

For the knight problem, the notion of setting up parity correctly will be given 0.5 points. The observation that the parity of the knight never changes will get 1 point, and putting everything together gets another 0.5 point.

Problem 3: Countable vs uncountable

(This problem is based on questions in Homework 7.)

(a) **(2 points)** Consider the set $\mathbb{B}$ of all **infinite** binary sequences that do not contain the pattern **11**. Use Cantor's diagonalization method to prove that $\mathbb{B}$ is uncountable. Explain clearly all the steps in your proof, including how you will precisely construct a sequence in $\mathbb{B}$ that will provide the contradiction.
Hint: Think of each infinite sequence as a sequence of pairs of bits.

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(b) **(2 points)** Define $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, where ∞ is a special element. First, explain why $\mathbb{N}^\infty$ is countable. Then, if the pattern in the above description of $\mathbb{B}$ is changed from **11** to **10**, prove that $\mathbb{B}$ becomes countable by providing a bijection with $\mathbb{N}^\infty$. (You don't have to rigorously prove that your function is a bijection as long as you explain it well.) Alternatively, you may describe how you can list sequences of $\mathbb{B}$ in such a way that each will have a finite rank.

Answers to Problem 3.

(a) Given $f : \mathbb{N} \to \mathbb{B}$ is a bijection (giving rise to a list of infinite sequences), we construct a sequence $w \in \mathbb{B}$ that does not show up in the list. To do that, we think of each infinite sequence as a sequence of pairs of bits, and we make the i^{th} pair of w different from the i^{th} pair of the sequence given by $f(i)$. This way, w is not in the list. To make sure $w \in \mathbb{B}$, we can adopt the following rules (they are not unique, there could be other rules):

$$00 \to 01$$

$$01 \to 00$$

$$10 \to 00$$

This way, all 1s in w are in even positions, so there will be no consecutive 1s in w. This provides the contradiction needed for Cantor's diagonalization proof.

(b) $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ is countable because it is the union of two countable sets: $\mathbb{N}$ is one of them, and $\{\infty\}$ (finite, with a single element) is the other.

Given that no **10** can show up in any of the infinite sequences, once a bit is 1, all subsequent bits must be 1. The infinite sequences look like this:

$0000\ldots$ (infinite sequence of 0s)
$1111\ldots$
$0111\ldots$
$0011\ldots$
$0001\ldots$
$\vdots$

We can define our bijection $f : \mathbb{B} \to \mathbb{N}^\infty$ as follows:

$$f(0000\ldots) = \infty$$

$$f(\underbrace{000\ldots1}_{k}111\ldots) = k$$

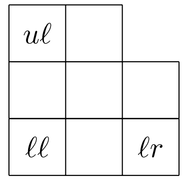
In other words, f maps each sequence to the position of its first 1 in the sequence, with the infinite sequence of 0s mapped to the special element ∞. One can also envision the following order in which every sequence has a finite rank:

$$0000\ldots, 1111\ldots, 0111\ldots, 0011\ldots, 0001\ldots, \ldots$$

Grading policy: For part (a), 0.5 points are given to the general setup of Cantor's diagonalization, and 0.5 points are given to each of the 3 rules.

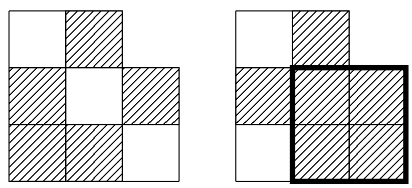
For part (b), 1 point is given to a correct explanation of why $\mathbb{N}^\infty$ is countable, and 1 point to the proof that $\mathbb{B}$ is countable by bijection with $\mathbb{N}^\infty$.

Problem 4: More inspired by Chess...
The following two questions are independent, but they both refer to the same board with 8 squares and three corners $u\ell$, $\ell\ell$, and ℓr, as shown below:



(This question is designed to check if you can use inclusion-exclusion to count, and is similar to some questions we have seen in class and on previous tests.)

(a) **(2 points)** Each square is to be colored either black or white. A coloring of the squares is good iff it does not create a 2×2 black square. For instance, the first coloring below is good because it does not create any 2×2 black square, but the second coloring is bad because it puts a 2×2 black square in the lower right corner.



good bad

How many good colorings are there? Find this number by counting the bad colorings using Inclusion-Exclusion. Let $S_{u\ell}$ be the set of colorings that put a 2×2 black square in the upper left corner. Define $S_{\ell\ell}$ and $S_{\ell r}$ similarly for the lower left corner and the lower right corner, respectively. Write a formula for $|S_{u\ell} \cup S_{\ell\ell} \cup S_{\ell r}|$, then use it to find the number of bad colorings, then figure out the number of good colorings.

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(This question is designed to check if you can identify a proper proof technique given a particular setup, it is similar in spirit to some questions that we have seen in class.)

(b) **(2 points)** We place 4 queens on the above board (each queen in its own square). Prove that two of them can attack each other by moving one square only. Recall that a queen can move in all 8 directions. You get some credit for identifying an appropriate proof technique (name it).

Answers to Problem 4.

(a) The inclusion-exclusion formula is:

$$|S_{ul} \cup S_{ll} \cup S_{lr}| = |S_{ul}| + |S_{ll}| + |S_{lr}| - |S_{ul} \cap S_{ll}| - |S_{ul} \cap S_{lr}| - |S_{ll} \cap S_{lr}| + |S_{ul} \cap S_{ll} \cap S_{lr}|$$

Having one 2x2 black square leaves 4 uncolored squares which can be colored in $2^4 = 16$ ways. Therefore,

$$|S_{ul}| = |S_{ll}| = |S_{lr}| = 16$$

Having one 2x2 black square in the upper left corner, and one 2x2 square in the lower left corner, leaves 2 uncolored squares which can be colored in $2^2 = 4$ ways. The same is true for the lower left corner and the lower right corner. Therefore,

$$|S_{ul} \cap S_{ll}| = |S_{ll} \cap S_{lr}| = 4$$

Having one 2x2 black square in the upper left corner, and one 2x2 square in the lower right corner, leaves 1 uncolored square which can be colored in $2^1 = 2$ ways. Therefore,

$$|S_{ul} \cap S_{lr}| = 2$$

Finally, having three 2x2 black squares leaves no squares to be colored. So,

$$|S_{ul} \cap S_{ll} \cap S_{lr}| = 1$$

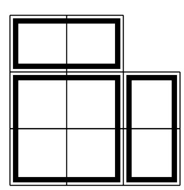
Putting things together, we get

$$|S_{ul} \cup S_{ll} \cup S_{lr}| = 16 + 16 + 16 - 4 - 4 - 2 + 1 = 39$$

This means that the number of good colorings is $2^8 - 39 = 256 - 39 = 217$.

(b) This is a great candidate for pigeonhole, because no matter how we place the 4 queens, the same consequence is guaranteed. We need to prove that 2 queens can attach each other by moving one square only. The language of pigeonhole suggests that we need to prove two objects fall in the same box. If the queens are the objects, then we need 3 boxes to make this happen.

There are many ways to "virtually" create the three boxes. Here's one way:



When we place 4 queens in 3 boxes, one box must contain at least 2. In each box, a queen can reach any square in one move. So when 2 queens fall in the same box, they can attach each other by moving one square only. Done.

Grading policy: For part (a), 1 point for a correct formula, another point for the rest of the counting setup. How this point is divided will be determined by the grader to achieve practicality and consistency.

For part (b), 1 point is given for identifying the pigeonhole technique, 0.5 points are given for a correct setup of objects and boxes, and 0.5 points are given for a correct finishing of the argument and connecting the pigeonhole world to the chess world.

Not used.

Feel free to provide a drawing, a poem, an idea, or any form of communication that conveys a connection between math and something else. This can be funny, serious, witty, sarcastic, mysterious, happy, sad, angry, politically correct or incorrect, short, long, or anything that you want it to be, your choice... I will select a few of the interesting ones to explore in a future lecture.